Consecutive sums
NMCC 2018

Mankkaa
school 8M
Espoo, Finland

Contents

1. Introduction

In this subject report we look into numbers that can be presented as sums of consecutive positive integers in one or more ways, as well as numbers that cannot be presented as a sum of these integers.

In addition to this, we investigated how randomly chosen numbers can be presented as the sums of consecutive positive integers in all possible ways.

In this subject report we do not consider the number one as an odd factor.

We used table sums in our solutions as well as general formulas that we derived from the arithmetic sum $(k+1)+(k+2)+\ldots+(k+n)$.

2. Solving the problem

We started our project on February 5th 2018 by reading the instructions. We split into small groups to solve the report's questions. The groups started working by tabulating the sums of consecutive numbers. Soon we started to get solutions and created a project folder in Google Drive, in which the class wrote notes and held a diary. In an another lesson Eemil, Leo and Henrik started preparing the exhibition. The programs that can calculate the sums in which a number can be presented, as well as what numbers can be written as sums in more than one way were written mainly by Leo, with the help of Lauri and Sam.

During the following lessons some students discussed the general formulas of the report's questions and some moved on to plan the exhibition. Inka designed the cover art of the report, and Juho developed formulas where the sums of consecutive numbers can be presented as the product of two integers.

We started working on the report on the 26th of February when Mimi, Olivia, Katri and Inka began writing the introduction.

At the beginning of the project, the whole class shared thoughts and findings with each other.

2.1 Numbers that can be presented as a sum of positive consecutive integers

We started solving the task by listing sums in a chart with a different number of addends.

We studied odd and even numbers separately.

Odd numbers

Every odd number is form $2n + 1$ and can be presented as $2n + 1 = n + (n + 1)$.

The number 137, for example:

$137 = 2n + 1 \quad || -1$

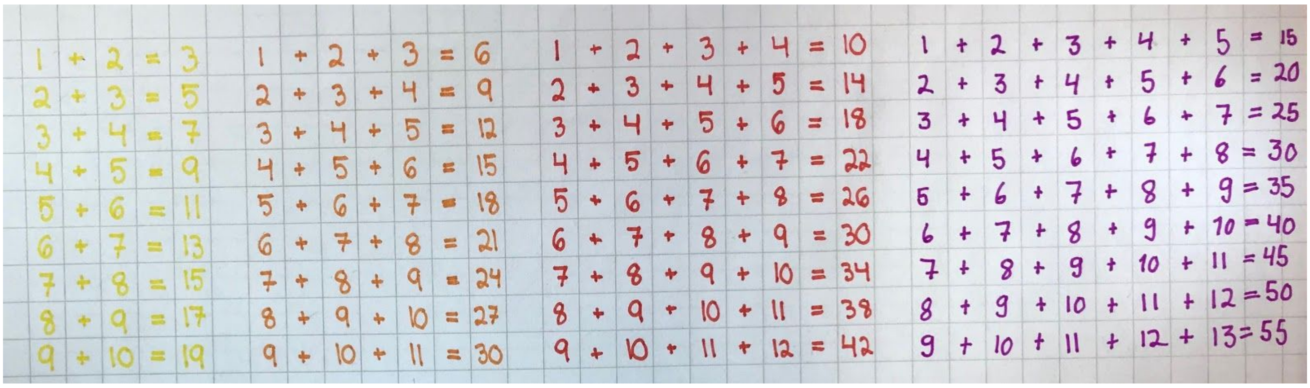
$136 = 2n \quad || : 2$

$68 = n$

$137 = n + (n + 1) \quad | \, n = 68$

$137 = 68 + (68 + 1)$

$137 = 68 + 69$



We made a chart that has some simple sums with two to five addends.

Even numbers

Even numbers that have odd factors can be written as sums of consecutive numbers.

$3n = (n-1) + n + (n+1)$

$5n = (n-2) + (n-1) + n + (n+1) + (n+2)$

Generally:

$(2k+1)\,n = (n-k) + (n-k+1) + \dots + (n-1) + n + (n+1) + \dots + (n+k)$

Examples

The number 24 can be presented as the product $3 \cdot 8$.

The odd factor is 3 and the n of the formula above is 8.

Therefore

$24 = (8-1) + 8 + (8+1)$

$24 = 7 + 8 + 9.$

The number 42 can be presented as a product $42 = 7 \cdot 6$.

The odd factor is 7 and the n of the formula above is 6.

Therefore

$42 = (6-3) + (6-2) + (6-1) + 6 + (6+1) + (6+2) + (6+3)$

$42 = 3 + 4 + 5 + 6 + 7 + 8 + 9.$

The number 14 can be presented as product $14 = 7 \cdot 2$.

The odd factor is 7 and the n of the formula above is 2.

Therefore

$14 = (2-3) + (2-2) + (2-1) + 2 + (2+1) + (2+2) + (2+3)$

$14 = -1 + 0 + 1 + 2 + 3 + 4 + 5$

In this case, the first addend is negative. However, the sum from -1 to 1 equals 0, so

$14 = 2 + 3 + 4 + 5$

We placed the numbers between 1 and 100 on a chart and highlighted the numbers that can be presented as sums of consecutive integers with red.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	43	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	84	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

We noticed that every highlighted number has at least one odd factor. The non-highlighted numbers $(1,\ 2,\ 4,\ 8, \dots)$ have no odd factors (other than the number one, of course).

We also investigated this with greater numbers and tested the factors using calculators.

$468 = 9 \cdot 52$ – the odd factor is 9.

The number can be presented as the sum $468 = 48 + 49 + 50 + 51 + 52 + 53 + 54 + 55 + 56$.

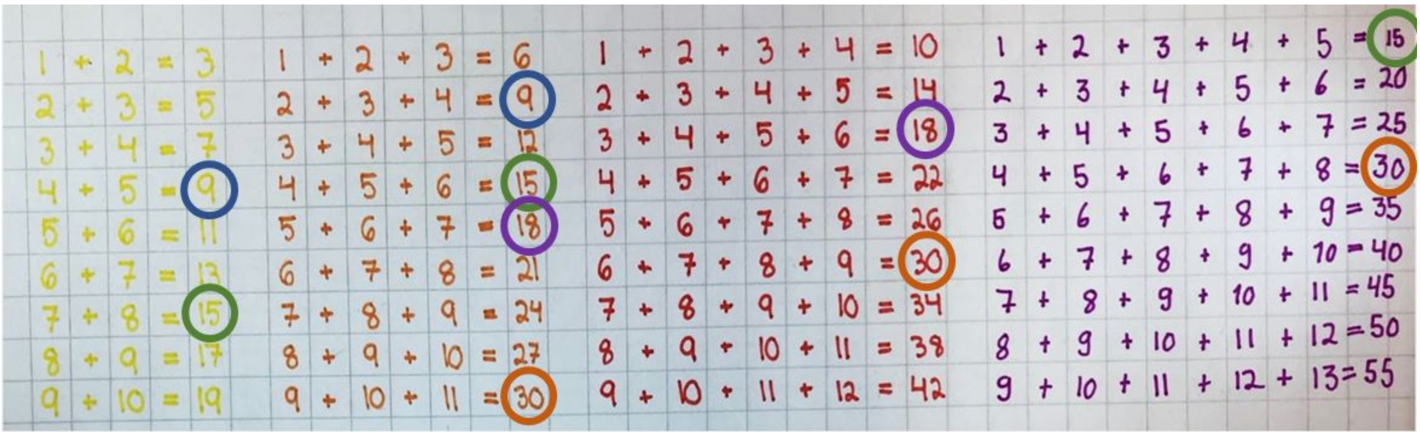
$6\,250 = 5 \cdot 1250$ – the odd factor is 5.

The number can be presented as the sum $6\,250 = 1\,248 + 1\,249 + 1\,250 + 1251 + 1252$.

Conclusion

All numbers that have at least one odd factor can be presented as sums of consecutive integers.

2.2 Numbers that can be written as a sum of positive consecutive integers in more than one way



We circled the numbers that appear on the chart more than once. Of them, the odd numbers 9 and 15 have at least one odd factor and the even numbers 18 and 30 have at least two odd factors.

As can be seen in the chart, the number 9 can be written as a sum in two different ways:

9 = 4 + 5

9 = 2 + 3 + 4

The number 15 can be written as a sum in three different ways:

15 = 7 + 8

15 = 4 + 5 + 6

15 = 1 + 2 + 3 + 4 + 5

We also noticed that prime numbers can only be written in one way. For example:

7 = 3 + 4

19 = 9 +10

From that we concluded that a number can be written in as many sums as it has odd factors:

The factors of **18** are 1, 2, **3**, 6, **9** and 18 so we can write it as $3 \cdot 6$ and $9 \cdot 2$

$18 = 5 + 6 + 7$

$18 = 3 + 4 + 5 + 6$

The factors of **30** are 1, 2, **3**, **5**, 6, 10, **15** and 30 so we can write it as $3 \cdot 10$, $5 \cdot 6$ and $15 \cdot 2$

$30 = 9 + 10 + 11$

$30 = 4 + 5 + 6 + 7 + 8$

$30 = 6 + 7 + 8 + 9$

The factors of **33** are 1, **3**, **11** and **33** so we can write it as $3 \cdot 11$, $11 \cdot 3$ and $33 \cdot 1$

$33 = 10 + 11 + 12$

$33 = 3 + 4 + 5 + 6 + 7 + 8$

$33 = 16 + 17$

Conclusion

Numbers that have two or more odd factors can be written as sums of consecutive integers in more than one way.

2.3 Numbers that cannot be presented as a sum of positive consecutive integers

When investigating the third question we made use of the chart presented in 2.1.
In the chart the numbers that cannot be written as sums are 1, 2, 4, 8, … 64.
We soon noticed they all are powers of number two.

We came to the conclusion that powers of number 2, i.e $\mathbf{2^n}$, $\mathbf{n} \in \mathbb{N}$ cannot be written as sums of consecutive positive integers.

We proved this by investigating the general sum of consecutive numbers.

May $(k + 1)$ be the first addend and n the number of addends in sum
$(k + 1) + (k + 2) + (k + 3) + \ldots + (k + n),\ k \geq 0,\ n \geq 2$

The sum can be presented in the form
$1 + 2 + 3 + \ldots + n + nk$
$= \frac{1+n}{2} \cdot n + nk$
$= n\,(\frac{1+n}{2} + k)$

Deriving the sum formula

Adding up two equal sums
$(1 + 2 + 3 + \ldots + n) + (n + (n - 1) + (n - 2) + \ldots + 1)$
$= (n + 1) + ((n - 1) + 2) + ((n - 2) + 3) + \ldots + (1 + n)$
$= (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1)$
$= (n + 1) \cdot n$
When this is divided by two, the formula of one sum results in
$\frac{n+1}{2} \cdot n$

1. If n is an even number, it is of the form $2m,\ m \geq 1$

$2m(\frac{1+2m}{2} + k)$

$= m(1 + 2m + 2k) = m(2(m + k) + 1)$

Because m and k are integers, $2(m + k)$ is even and $2(m + k) + 1$ odd.

2. If n is an odd number, it is of the form $2m + 1,\ m \geq 0$

$(2m + 1)(\frac{1+2m+1}{2} + k)$

$= (2m + 1)(\frac{2m+2}{2} + k)$

$= (2m + 1)(m + k + 1)$

Because m and k are integers, $2m + 1$ is odd.

Conclusion

When the sum is presented as the product of two numbers, at least one of the factors needs to be odd. In the powers of number two, all the factors are even so they cannot be presented as such sums.

2.4 Presenting an arbitrary number as different sums of positive consecutive integers

In this chapter we research how an arbitrarily chosen number can be presented in every possible way as sums of positive consecutive integers.

As said before, a sum of consecutive integers can be presented as the product

$m(2m + 2k + 1)$, in which the number of addends $n = 2m$

or

$(2m + 1)(m + k + 1)$, in which the number of addends $n = 2m + 1$

The researched number is presented as all possible products of two factors in which at least one of the factors is an odd number deviating from the number one.

Example 1

The number 2018 can be presented as product explained above only in the form

$2018 = 2 \cdot 1009$.

The formula is form $m(1 + 2m + 2k)$, in which $m = 2$

$m(1 + 2m + 2k) = 2018 \qquad | \, m = 2$

$2(1 + 4 + 2k) = 2018$

$10 + 4k = 2018 \qquad || - 10$

$4k = 2008 \qquad || : 4$

$k = 502$

The first term of the sum is $k + 1 = 502 + 1 = 503$

There are $2m = 2 \cdot 2 = 4$ addends

$2018 = 503 + 504 + 505 + 506$

If the number of addends is chosen to be an odd number $n = (2m + 1)$, so $2m + 1 = 1009$, of which $m = 504$, then the formula $(2m + 1)(m + k + 1)$ is used.

$(2m + 1)(m + k + 1) = 2018 \qquad | \; m = 504$

$(2 \cdot 504 + 1)(504 + k + 1) = 2018$

$1009\,(505 + k) = 2018 \qquad || : 1009$

$505 + k = 2 \qquad || - 505$

$k = \; - 503$

The first addend is $k + 1 = \; - 503 + 1 = \; - 502$ and the number of them is 1009. As the sum forms of consecutive integers, the value of the last addend is $- 502 + 1008 = 506$.

Then the total sum must be

$- 502 + (- 501) + \ldots + (- 1) + 0 + 1 + \ldots + 501 + 502 + 503 + 504 + 505 + 506$.

Because the sum from the beginning to 502 is 0, the whole sum gets simplified to form that is the same as before.

So the number 2018 can be presented as the sum of positive consecutive numbers in only one way.

Example 2

Number 365 can be presented in products that have an odd factor in three ways:

$365 = 1 \cdot 365 = 5 \cdot 73 = 73 \cdot 5$

Finding different sums using the products:

The form of the product $\mathbf{1 \cdot 365}$ is $m(1 + 2m + 2k)$, in which $m = 1$

$m(1 + 2m + 2k) = 365 \qquad | \; m = 1$

$1(1 + 2 + 2k) = 365$

$2k + 3 = 365$

$2k = 362$

$k = 181$

The first addend $k + 1 = 181 + 1 = 182$ and there are

$2m = 2 \cdot 1 = 2$ addends.

$365 = 182 + 183$

The form of the product **5 · 73** is $(2m+1)(m+k+1)$ $\quad | (2m+1)=5$, so $m=2$

$5(2+k+1)=365$

$5(3+k)=365$

$15+5k=365 \quad || -15$

$5k=350 \quad || : 5$

$k=70$

The first addend $k+1=70+1=71$ and there are $2m+1=2\cdot 2+1=5$ addends.

$365=71+72+73+74+75$

If the other formula was used, the simplified sum would still remain the same.

The form of the product **73 · 5** is $(2m+1)(m+k+1)$ $\quad | 2m+1=73$, so $m=36$

$73\cdot(36+k+1)=365 \quad || : 73$

$37+k=5 \quad || -37$

$k=-32$

The first addends $k+1=-32+1=-31$ and there are $2m+1=73$ addends . The last addend equals to $-31+72=41$.

$365=-31+(-30)+\ldots+31+32+\ldots+40+41.$

The sum can be simplified to the form $365=32+33+34+\ldots+41$.

If the other formula was used, the same sums would still form. The formula should be chosen so that $k\geq 0$. That way, the neutralizing of negative numbers and their opposite numbers can be avoided.

Conclusion

An arbitrarily chosen number can be presented as all possible sums of positive consecutive numbers by finding all the products of two factors in which at least one factor is odd. Then, using the previous formulas, the number of the addends and the value of the first addend is determined.

3. Summary

Our task was to solve questions related to the sums of consecutive positive integers. The assignment brought the 20 brains of our class together around one problem. Our conclusion was that a number can be written in as many forms as it has odd factors other than the number one. So only the powers of the number two cannot be written as such sums, as they have no odd factors besides the number one. All possible sums a number can be presented as can be found using general formulas. We wanted to avoid repeating the phrase“odd factors other than the number one” over and over, so we decided not to count the number one as an odd factor.

In addition to learning about solving and writing complex math we also learned a lot about our strengths and weaknesses. We have a unique group of people ranging from athletes, class clowns and slackers to university level mathematicians.

Juho had created formulas to determine the sums from the products. He explained how they worked but understanding the overall picture was challenging for some of us. Gradually the rest of the class began to understand the formulas when they were applied in practice. We used the general formulas that Juho made to prove the first questions and to solve our own question. We learned that the formulas are the most practical calculating method.

We used different working methods during the project. When figuring out the sums, we started by making tables. It was a natural thing to do as it was tangible and easy to understand. Leo's approach was different: he wrote computer programs that solved the same questions faster and more efficiently while also working with large numbers. We soon agreed to use them, but we were not sure about the reliability of the programs and that turned out to be a small problem. Were there sums that the computer could not find?

We learned that general formulas are superior to numerical examples. Problems occurred when many people edited the report at the same time. The material kept shifting around and getting all messed up. In addition, the report became too long, so we had to shorten it. We had some challenges with organizing so we used a website in which everyone could see each other’s tasks.

As the deadline drew nearer, we split into small workgroups because working was confusing in larger groups. We got better perspectives to our own ideas from each other. Mario, Sam and Luukas kept a diary about what happened during working. In the early stages, Vilho, Ikko and Aapo worked on the report and later Aurora, Ella K. and Ella E. joined them. Agda and Sanni also worked on the exhibition, but mainly on the report.

Our teachers Johannes and Sami gave us all of the responsibility in making the project. However, they helped us to write Juho's formulas in a way everyone could understand. We came to the conclusion that our class could accomplish anything together as long as we can concentrate on it.

Report done, yay!