Sums of Consecutive Integers

NMCC 2018,

Tranby School, Class 9D

Buskerud, Norway

Contents:

Introduction

We interpreted the task, as we were to focus on finding the most numbers that can be written as sums of consecutive integers, finding the patterns behind this and finding out which numbers cannot be written as a sum of consecutive integers.

Furthermore, the task states we had to ask ourselves at least one question in addition to the one we already have to answer.

The class was divided into groups of four-five people, and these groups wrote logs for use when constructing the report.

Mathematical process

During our process of working with this task, our class utilized different kinds of methods to reach our results. In the following section, we describe these methods and how we utilized them to find our answers.

Chaotic trying

In the beginning of this process, we started out writing all the numbers of consecutive integers we found in an unsystematic way. All the groups experimented on which method to use. We started using tools to make this more approach systematic. The groups started out only using a calculator. Later our teacher suggested we should also use other digital tools.

Table

The logs became quite chaotic, as our numbers and sums were written in no specific order. Therefore we made a table where we wrote the numbers from one to fifty, and which of these could be written as sums of two, three, four and five consecutive integers. This turned out to be much more effective, and we started to see patterns between the number of consecutive integers, and the gaps between the numbers which could be written as a sum of the same number of consecutive integers.

	2	3	4	5	6	7	8	9
1	0+1							
2								
3	1+2	0+1+2						
4								
5	2+3							
6		1+2+3	0+1+2+3					
7	3+4							
8								
9	4+5	2+3+4						
10			1+2+3+4	0+1+2+3+4				
11	5+6							
12		3+4+5						
13	6+7							
14			2+3+4+5					
15	7+8	4+5+6		1+2+3+4+5	0+1+2+3+4+5			
16								
17	8+9							
18		5+6+7	3+4+5+6					
19	9+10							
20				2+3+4+5+6				
21	10+11	6+7+8			1+2+3+4+5+6	0+1+2+3+4+5+6		
22			4+5+6+7					
23	11+12							
24		7+8+9						
25	12+13			3+4+5+6+7				
26			5+6+7+8					
27	13+14	8+9+10			2+3+4+5+6+7			
28						1+2+3+4+5+6+7	0+1+2+3+4+5+6+7	
29	14+15							
30		9+10+11	6+7+8+9	4+5+6+7+8				
31	15+16							
32								
33	16+17	10+11+12			3+4+5+6+7+8			
34			7+8+9+10					
35	17+18			5+6+7+8+9		2+3+4+5+6+7+8		
36		11+12+13					1+2+3+4+5+6+7+8	0+1+2+3+4+5+6+7+8
37	18+19							
38			8+9+10+11					
39	19+20	12+13+14			4+5+6+7+8+9			
40				6+7+8+9+10				

Fig. 1

Excel

We constructed this spreadsheet (*Fig. 2*) by making column of consecutive integers and adding these together. All the integers are in column A, so in column B we added A2+A3, 1+2, thus getting the answer 3 in cell B2. We then added A3+A4 in the cell underneath, and so on and so forth. Then we did the same in column C, but this time we added A2+A3+A4. We continued to do this until the spreadsheet had a respectable size, and then we made an additional table at the side, showing how many ways each number can be written as a sum of consecutive integers.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	0	Two numb	Thre numb	Four numb	Five numb	Six numbe	Seven num	Eight numb	Nine numb	Ten numbe	Eleven nur	Twelve nur	Thirteen nu	Fourteen n	Fifteen nur	Sixteen nu	Seventeer
2	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153
3	2	5	9	14	20	27	35	44	54	65	77	90	104	119	135	152	170
4	3	7	12	18	25	33	42	52	63	75	88	102	117	133	150	168	187
5	4	9	15	22	30	39	49	60	72	85	99	114	130	147	165	184	204
6	5	11	18	26	35	45	56	68	81	95	110	126	143	161	180	200	221
7	6	13	21	30	40	51	63	76	90	105	121	138	156	175	195	216	238
8	7	15	24	34	45	57	70	84	99	115	132	150	169	189	210	232	255
9	8	17	27	38	50	63	77	92	108	125	143	162	182	203	225	248	272
10	9	19	30	42	55	69	84	100	117	135	154	174	195	217	240	264	289
11	10	21	33	46	60	75	91	108	126	145	165	186	208	231	255	280	306
12	11	23	36	50	65	81	98	116	135	155	176	198	221	245	270	296	323
13	12	25	39	54	70	87	105	124	144	165	187	210	234	259	285	312	340
14	13	27	42	58	75	93	112	132	153	175	198	222	247	273	300	328	357
15	14	29	45	62	80	99	119	140	162	185	209	234	260	287	315	344	374
16	15	31	48	66	85	105	126	148	171	195	220	246	273	301	330	360	391
17	16	33	51	70	90	111	133	156	180	205	231	258	286	315	345	376	408
18	17	35	54	74	95	117	140	164	189	215	242	270	299	329	360	392	425
19	18	37	57	78	100	123	147	172	198	225	253	282	312	343	375	408	442
20	19	39	60	82	105	129	154	180	207	235	264	294	325	357	390	424	459
21	20	41	63	86	110	135	161	188	216	245	275	306	338	371	405	440	476
22	21	43	66	90	115	141	168	196	225	255	286	318	351	385	420	456	493
23	22	45	69	94	120	147	175	204	234	265	297	330	364	399	435	472	510
24	23	47	72	98	125	153	182	212	243	275	308	342	377	413	450	488	527
25	24	49	75	102	130	159	189	220	252	285	319	354	390	427	465	504	544
26	25	51	78	106	135	165	196	228	261	295	330	366	403	441	480	520	561
27	26	53	81	110	140	171	203	236	270	305	341	378	416	455	495	536	578
28	27	55	84	114	145	177	210	244	279	315	352	390	429	469	510	552	595
29	28	57	87	118	150	183	217	252	288	325	363	402	442	483	525	568	612
30	29	59	90	122	155	189	224	260	297	335	374	414	455	497	540	584	629
31	30	61	93	126	160	195	231	268	306	345	385	426	468	511	555	600	646
32	31	63	96	130	165	201	238	276	315	355	396	438	481	525	570	616	663
33	32	65	99	134	170	207	245	284	324	365	407	450	494	539	585	632	680
34	33	67	102	138	175	213	252	292	333	375	418	462	507	553	600	648	697
35	34	69	105	142	180	219	259	300	342	385	429	474	520	567	615	664	714

Fig. 2

To illustrate this, we used colour coding. The colour tells us about how many times the number can be written as sums of consecutive integers. The numbers in pink can be written as sums of consecutive integers one time, the numbers in purple two times, turquoise three, orange four, yellow five light green six, light blue seven and dark blue nine times.

Triangular numbers

Our teacher brought plastic pieces to the classroom, which some students laid out underneath each other (See *Fig. 3*), with one additional piece in each row. When we looked at the figure below, we saw similarities between triangular numbers and sums of consecutive integers.

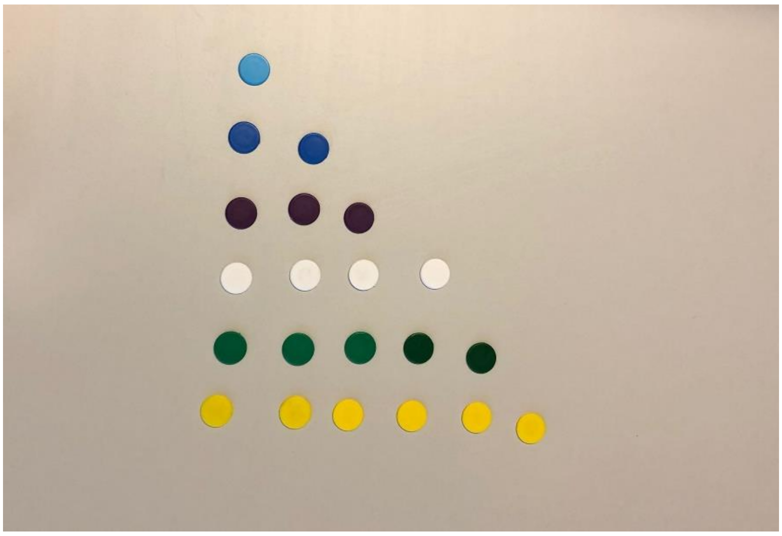

Fig. 3

Supplementary questions we asked ourselves

We chose to ask ourselves; what is the pattern between the numbers that cannot be written as a sum of consecutive integers? Moreover, what happens if we use negative numbers in addition to the positive integers? The curiosity around negative numbers emerged when some of our class started to experiment with consequences of using numbers below zero.

Results

Negative numbers

As mentioned one of the supplementary questions we asked ourselves was what would happen if we used negative numbers in addition to positive. Thus it is possible to write every single number as a sum of consecutive integers, even negative numbers.

$$-3 + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 = 4$$

$$\big((-3) + (-2) + (-1) + 0 + 1 + 2 + 3\big) + 4 = 0 + 4 = 4$$

We see that by writing the same amount of negative and positive integers the majority of the numbers equalize themselves and becomes zero. The next number added will be the sum of these numbers. Thus it is possible to write every single number as a sum of consecutive integers, even negative numbers.

Patterns in numbers that cannot be written as sums of consecutive positive integers

In addition we wondered which numbers could not be written as a sum of consecutive positive integers. We saw early on that some numbers, such as 4 and 16, could not be written as a sum of consecutive integers. When we organized our numbers, we quickly saw the pattern. As the row of numbers is 2, 4, 8, 16, 32 etc. we have to double the current number in order to find the next. This would be equivalent to multiplying each number by two. As every one of these numbers can be written as two times itself n times, they can all be expressed as 2^n.

Patterns in numbers that can be written as sums of consecutive positive integers

During the process of organizing the sums, we also decided to search for patterns. We quickly noticed that the gap between the numbers that can be written as a sum of consecutive integers is the same as the amount of integers added. For example 1+2=3, and 2+3=5. We discovered that if we add two integers, the difference between the sums is two. Later in our work, we thought it was appropriate to create a formula in order to calculate a sum with ease. This is the result:

$$\sum_{k=n}^{n+(x-1)} k$$

The sum of k, where k goes from n to n+(x-1), n is the first number to be added, and x is the number of integers to be added. By using this formula, we only have to put in two numbers, and then we instantly get the sum of whichever consecutive integers we want.

Odd numbers

As we discovered which numbers could be written as sums of consecutive integers, we pursued to find out why every odd number can be written as a sum of consecutive integers. Besides that, most of the odd numbers can be written as sums of three, four or more consecutive integers. The reason all odd numbers can be written as a sum of consecutive integers is a result caused by two consecutive integers, where every sum is an odd number. Two consecutive integers will always result in an odd number. The reason is that two consecutive integers will always include one even number and one odd number. When we add these together, we get an odd number. When we take this information in addition to that each sum of two consecutive integers increases with two each time, we have enough information to conclude that all odd numbers can be written as a sum of consecutive integers.

Below is the formula we made to generalize how every odd number can be written as a sum of consecutive integers, inspired by how every odd number (represented in the formula as "n") can be written as sums of two consecutive integers. This is the formula:

$$\left(\frac{n}{2} - 0{,}5\right) + \left(\frac{n}{2} + 0{,}5\right)$$

Even numbers

While searching for patterns on how every even number could be written as a sum of consecutive integers, we found a consistent pattern with the numbers that can be written as three or four consecutive integers.

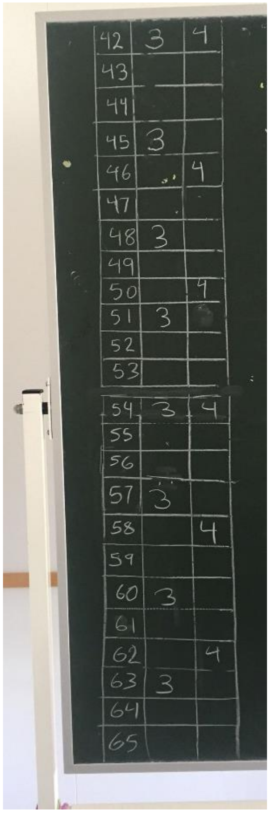

Fig. 4

Fig. 4 shows how sums of three and four consecutive integers creates a pattern that repeats itself every twelfth number. Six of the numbers can be written as a sum of three or four numbers, and six of them cannot be written like this. If we exclude the numbers that can be written as sums of two consecutive integers, since we already found a pattern for odd numbers, we are only left with two even numbers that cannot be written as three or four consecutive integers every twelfth number. These numbers can either be written as sums of different numbers of consecutive integers, or are binary and cannot be written as a sum of only positive integers.

Prime numbers

All prime numbers, except two, can be written as sums of consecutive integers. This is primarily due to the fact that every prime number after two is an odd number. Since the prime numbers after two can be written as a sum of consecutive integers and are odd numbers, we can use the same explanation as we used to explain why odd numbers can be written as sums of consecutive integers in “Odd numbers” above, to explain why the prime numbers over two can be written as sums of consecutive integers.

Every prime number, except two, can be written as a sum of consecutive positive integers one time, because the prime numbers can only be divided by themselves and one. The connection behind the amount of odd divisors and the times a number can be written as a sum of consecutive integers is written under “Numbers of factors”.

The formula we use for finding all prime numbers (excluding two, because it cannot be written with only positive numbers) is the same as the formula we use for odd numbers, because all the prime numbers over three are odd numbers. This is the formula:

$$\left(\frac{n}{2} - 0{,}5\right) + \left(\frac{n}{2} + 0{,}5\right)$$

Triangular numbers

Triangular numbers are the numbers where the space between the numbers increase by one for each number and can be illustrated in a triangular pattern. The triangular numbers up to 100 are the following:

1 3 6 10 15 21 28 36 45 55 66 78 91

If we use a picture of the figures the class made in addition to some colour coding, it is easier to see the pattern between these numbers.

Fig. 5

1 2 3 3 6 4 10 5 15 6 21 7 28 8 36 9 45 10 55 11 66 12 78 13 91

The gap between the numbers consistently rise by one each time. This fits the system of the sums of consecutive integers where the first number in the sequence of numbers is one.

$$1(1)$$

$$3(1+2)$$

$$6(1+2+3)$$

$$10(1+2+3+4)$$

$$15(1+2+3+4+5)$$

$$21(1+2+3+4+5+6)$$

$$28(1+2+3+4+5+6+7)$$

$$36(1+2+3+4+5+6+7+8)$$

$$45(1+2+3+4+5+6+7+8+9)$$

$$55(1+2+3+4+5+6+7+8+9+10)$$

$$66(1+2+3+4+5+6+7+8+9+10+11)$$

$$78(1+2+3+4+5+6+7+8+9+10+11+12)$$

$$91(1+2+3+4+5+6+7+8+9+10+11+12+13)$$

As the illustration shows, every triangular number can be written as a given sum of consecutive integers. Funnily enough, the illustration looks like a triangle, this because the last number you add together is the same as numbers of consecutive integers.

Sum of the digits

Numbers which have digits that when added together make up the number 3, 6 or 9 can be written as sums of consecutive integers. In this formula, a= a number with a sum of the digits that equals three, six or nine:

$$a \div 3 = b$$

$$(b - 1) + b + (b + 1)$$

As an example, we use 3987. To prove that 3987 has a sum of the digits equal to three, six or nine, we add the digits together like this.

$$3 + 9 + 8 + 7 = 27$$

$$2 + 7 = 9$$

If the first sum of the digits is a number with more than one digit, you add the new digits together, and continue until you have a number with only one digit. Therefore, the sum of the digits of 3987 is 9 (and not 27).

Now we try the formula with our number:

$$3987 \div 3 = 1329$$

$$1329 - 1 + 1329 + 1329 + 1$$

$$1328 + 1329 + 1330$$

As we proved with our number, where the sum of the digits is three, six or nine, and our formula, every number where the sum of digits is three, six or nine, can be written as a sum of consecutive integers.

Multiplication table of three

We were trying to find a formula that shows how every number can be written as consecutive numbers. We did not manage to create a formula that worked for absolutely every number, but we manage to create a formula that works for the numbers in the multiplication table of three, five, and seven (because of this, the next two paragraphs are strongly related to this)

N=A number in the multiplication table of three, five or seven

$$S = \frac{N}{M}$$

M=Multiplication table of the number we have chosen

Fig. 6

Fig. 7

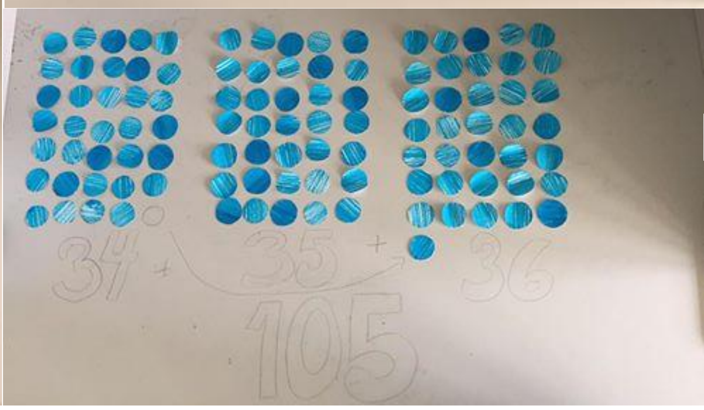

Fig. 8

This is the first formula we created to find out how every number in the multiplication table of three can be written as a sum of three consecutive numbers. We divided our starting number by three to find three equal numbers. To make these consecutive, we subtracted one from the first number, and added one to the last one. Since this formula was the first we made, it is not as sophisticated as the others we made. You can clearly see all the different parts in the formula.

$$N = (S - 1) + S + (S + 1)$$

Unfortunately this formula can only be used with the numbers in the multiplication table of three. See Fig.6 to 8 for illustration. The one below is more sophisticated, and can be used with the multiplication table of five and seven in addition to three. We will try to explain the function of every part of the formula. N is the sum we start with. It has to be a number in the multiplication table of three, five or seven. We divide N by M, and get M equal sums which is written as S. To get M consecutive integers, we add all the numbers from $S - \frac{M+1}{2}$ (we do this to get the lowest consecutive integer) to $S + \frac{M-1}{2}$ (we do this to get the highest consecutive integer). The sigma adds all the numbers from the lowest consecutive integer to the highest consecutive integer. The sum of every part is always equal to N.

$$N = MS = \sum_{k=S-\frac{M-1}{2}}^{S+\frac{M-1}{2}} k = N$$

The formula underneath is a formula combined with both the less sophisticated and the sophisticated formula. As always, every part is the equal to N.

$$N = MS = \sum_{k=S-\frac{M-1}{2}}^{S+\frac{M-1}{2}} k = (\mathrm{S} - 1) + \mathrm{S} + (\mathrm{S} + 1)$$

Multiplication table of five

The formula below follows the same principle as the less sophisticated formula for the multiplication table of three. It uses five numbers instead of three. See Fig. 6, 9 and 10.

Fig. 6

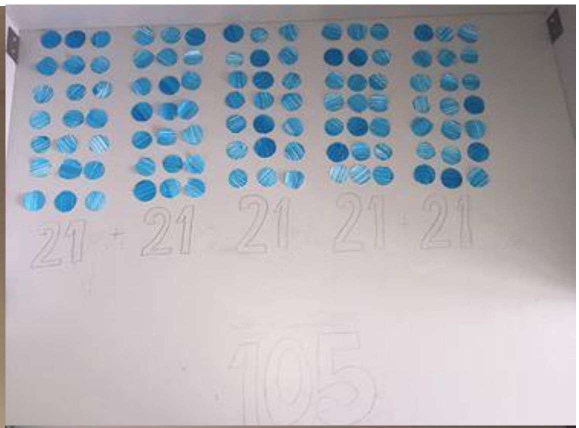

Fig. 9

Fig. 10

$$N = (S-2) + (S-1) + S + (S+1) + (S+2)$$

As mentioned, the same applies to the exact same sophisticated formula as the multiplication table of three. This is possible because the formula is made to fit the purpose of finding consecutive integers in the multiplication table of three, five and seven. The explanation of the formula can be found under Multiplication table of three.

$$N = MS = \sum_{k=\frac{M-1}{2}}^{S+\frac{M-1}{2}} k = N$$

Below we combined the less complicated and the sophisticated formula

$$N = MS = \sum_{k=\frac{M-1}{2}}^{S+\frac{M-1}{2}} k = (S-2) + (S-1) + S + (S+1) + (S+2)$$

Multiplication table of seven

Below is the less sophisticated formula for the multiplication table of seven.

$$N = (S-3) + (S-2) + (S-1) + S + (S+1) + (S+2) + (S+3)$$

For illustration, see Fig. 6, 11 and 12.

Fig. 11

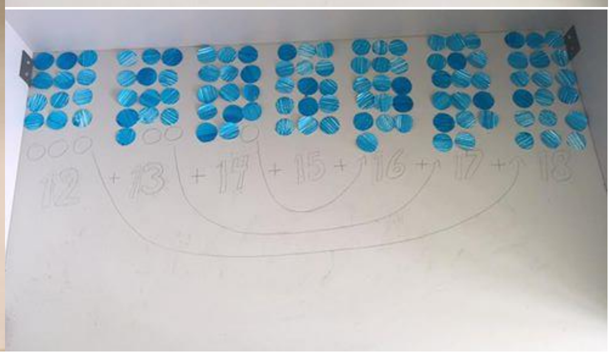

Fig. 12

Fig. 6

This is the exactly same formula that we made to find out how every number in the multiplication table of three, five and seven.

$$N = MS = \sum_{k=S-\frac{M-1}{2}}^{S+\frac{M-1}{2}} k$$

This is the combined formula:

$$N = MS = \sum_{k=S-\frac{M-1}{2}}^{S+\frac{M-1}{2}} k$$

$$= (S-3) + (S-2) + (S-1) + S + (S+1) + (S+2)$$

Numbers of factors

When we were searching for sums of consecutive integers on the internet, we came across an article about polite numbers. Unfortunately, this was a Wikipedia page (see Literature). In this article, we found some information about odd factors and sums of consecutive integers.

A discovery we made there was how many different ways any given number can be written as a sum of consecutive integers, was that the number of factors that each number had only included odd numbers, including itself. If we take 45 as an example since it is the first number that can be written as five sums of consecutive integers:

45-15-9-5-3

As we see, 45 has five factors including itself. If we compare our results here with our colour coded Excel-table (see Fig. 2), we see that 45 is yellow, the colour of five sums.

Let us try another number like 28.

28-14-7-4

28 can be factorised into 28, 14, 7, 4 and 2. This is four numbers. If this were our answer, our theory would be proved wrong, but if we now remove the even numbers, we are left with seven, the only odd number, because 28 can only be written as a sum of consecutive integers once.

This method can be used to find out how many times absolutely every positive integer can be written as a sum of consecutive integers.

Conclusion

The discoveries we made during our work with the task are the following:

We concluded that *not* all numbers are sums of positive consecutive integers, but if we include negative numbers, every number can be written as such sums.

The binary numbers are the numbers that cannot be written as sums of consecutive integers, but all the other numbers can be written at least one way.

We found a pattern with the sums of three and four consecutive integers that only leaves out two even numbers every twelfth number.

All the odd and prime numbers from three can be written as sums of consecutive integers with a specific and simple formula (the formula can work with the number 1 if we include 0).

The triangular numbers are the same as the numbers that can be written as consecutive integers where the first number in the sequence of numbers is one.

Numbers with a sum of the digits equal to three, six or nine, can be written as a sum of consecutive integers with the same formula.

We can write the numbers in the multiplication tables of three, five and seven, as sums of consecutive integers with one formula.

You can find out how many times every number can be written as a sum of consecutive integers by factorising it and counting how many factors the number have that is an odd number.

What this task has taught us

This task has taught us primarily to work together, but also a lot about how to present the results we have found, and how to discover things on our own since it was little (to no) information on the internet on the subject. Secondly, the class found it interesting to work with a project task in the math classes. We have also learned many things about consecutive integers and how to solve mathematical problems from the bottom. Finally, we have of course developed our mathematical capabilities and language through this task.

Literature

The only usage of other sources than ourselves was under "Numbers of factors" where we found inspiration on the Wikipedia page about polite numbers.

https://en.wikipedia.org/wiki/Polite_number

Read and found: 18.05.2018