NMCC/UngeAbel

2022

Tetraedre

Norway

Birralee International School

Year 9

Table of Contents

Part 1: Introduction and how we worked

Introduction: How did we interpret the problem/assignment?

In this assignment, we were tasked to investigate fractals, particularly tetrahedrons. The problem asks us to find the *n-th* term and term-to-term rule for the number of spheres and sticks in different sizes of tetrahedrons and to use different methods to compare and represent the information. As the problem requires understanding fractals, the class decided to make visual representations with magnetic spheres and sticks, graphs, models and self-made computer programs.

What is a fractal?

A fractal is a recurring shape or pattern that is often found in nature, geometrical and algebraic shapes (these are constructed by equations being solved repeatedly). Therefore, at any level of magnification, the pattern will look identical. Some of the most well-known fractals include Sierpinski's triangle (*Image #1*), the Fibonacci sequence (which can be used to describe the golden ratio) *(golden ratio: Image #2)* and the Mandelbrot set (*Image #3*). A fractal's recurring shape results in it having an indefinite area and perimeter.

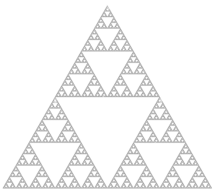


Image #1

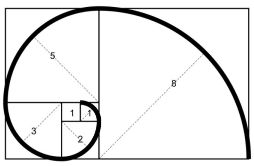


Image #2

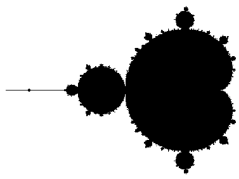


Image #3

Additionally, fractals are useful to understand difficult concepts, like how bacteria grow, the variation of a human's heartbeat, and how cell phones catch signals.

How did the class work together?

First, the class took a vote on three people who would be responsible for the report, display and poster. They were responsible for organising the entire class and making sure everything was on target to meet the deadline. Each student was assigned to a group, and while some began to make their own set of formulas, others had to report the class' discoveries. We finally found four formulas, which were tested. The n-th term and the term-to-term rules were found for both the sticks and spheres for the 3D Sierpinski's Triangle. Later on, the class had to decide on a theme for the overall display. After a long time of discussing and finding potential ideas of what it could be, we finally came to a conclusion, of having a Mandelbrot, fractals in nature and anatomy-themed display. The three leaders organised new groups for designing displays, finishing the report, and creating a presentation. These groups were created so each student could work on something they felt passionate about. Nonetheless, the class got to work on a fun and experimental mathematics competition.

How did the class overcome any challenges?

During the assignment, we faced multiple challenges including time management, procrastination and a lack of knowledge about the subject. Therefore, we solved most of the problems by discussing them in groups, sharing anything we learned with each other and utilising guidance from our teachers and students from Year 10 (previous UngeAbel Nordic Final winners). To increase efficiency, the three representatives created a plan for the class every day. The plan included journaling the progression of the groups in lesson time and discussing ideas about various aspects of the project. School events, like Dark day and Ski day, also limited the use of any electronic devices and reduced the class' working time.

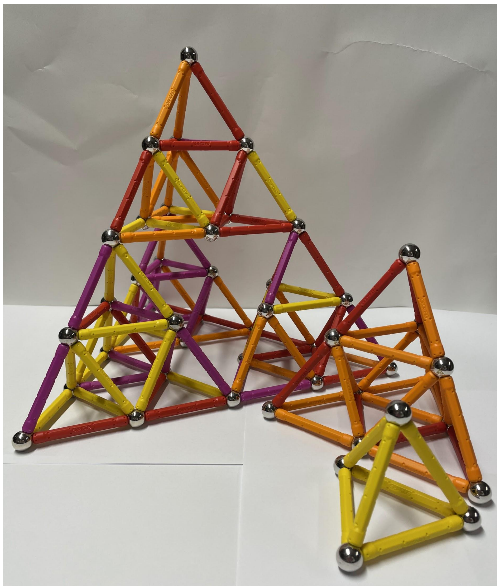
The use of sources, resources and information

Our class used a range of secondary sources, such as books from our school library, YouTube videos and websites to increase our knowledge of fractals. Primary sources such as our teachers and peers also came in handy; helping to strengthen our knowledge. Other primary resources such as Geogebra, Repl.it, Geomag sets, and spreadsheets were used to make various representations of our mathematical findings.

Part 2: Our mathematical process, solutions and discoveries

The Mathematical Process

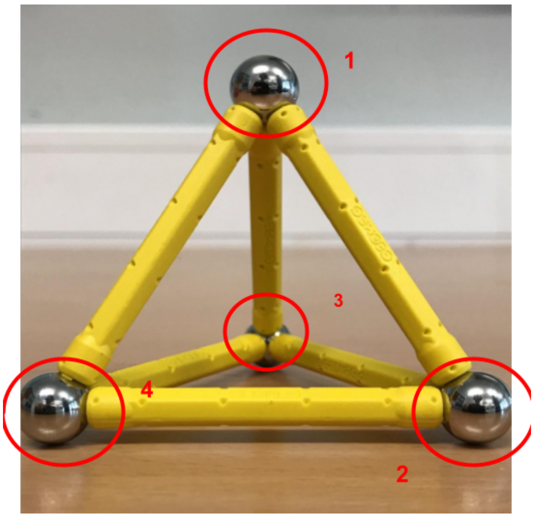
We were assigned to small groups and everyone started building tetrahedrons with spheres and sticks from the Geomag set the school ordered (magnetic sticks and spheres) weeks before the assignment.



Every group managed to make sizes 1 and 2 but had to collaborate to make size 3. For every size made, we recorded the number of sticks and spheres in graphs and searched for patterns.

Sizes 1, 2 and 3

Steps to finding the term-to-term rule

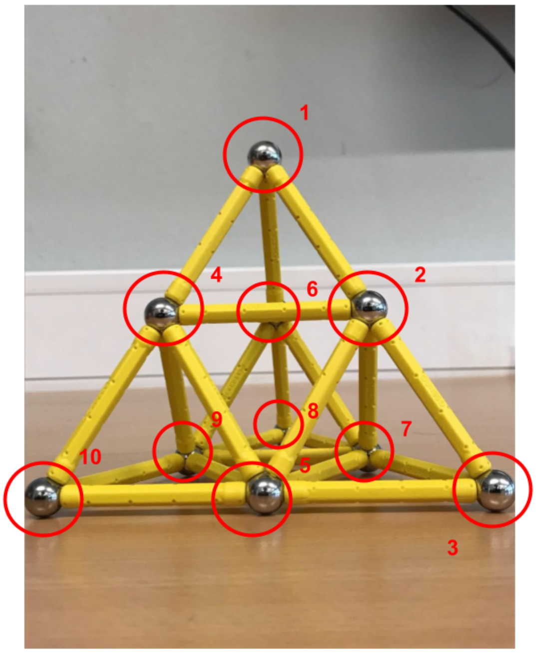


Size 1

Size 1. It has:

4 x *spheres*

6 x *sticks*

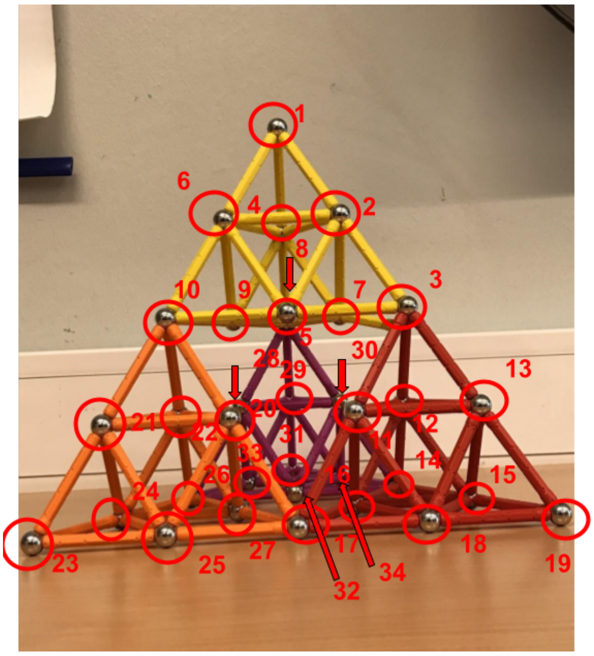


Size 2

Size two. It has:

10 x *spheres*

24 x *sticks*



Size 3

Size 3. It has:

34 x *spheres*

96 x *sticks*

Table representing the term-to-term rule:

Size (n)	1	2	3	4	5	6	7	Term-to-term rule
# of Spheres	4	10	34	*130*	*514*	*2050*	*8194*	**x4-6**
# of Sticks	6	24	96	*384*	*1536*	*6144*	*24576*	**x4**

The 4 is multiplied in both rules, since each size is made up of 4 tetrahedrons attached to each other. The minus 6 comes from the 6 points where the individual tetrahedron spheres connect.

There were some exciting patterns discovered in the graphs made, and one of them was that the last digit of the number of sticks from each size always alternated between six and four (**6,** 2**4**, 9**6**, 38**4**, 153**6** etc…), and the number of spheres always alternated between zero and four (**4**, 1**0**, 3**4**, 13**0**, 51**4** etc…). We also discovered that the number of spheres is every second binary number (**2**, 4, **8**, 16, **32**, 64, **128**) plus two (**2**+2 = *4*, **8**+2 = *10*, **32**+2 = *34*, **128**+2 = *130*, etc.). These patterns are connected to the concept of fractals, because they repeat themselves (every two last digits, the term-to-term rules, etc).

The *n*-th Term

The n-th Term: Sticks

After finding the term-to-term rule, different groups collaborated to determine the *n*-th term. It is a useful formula that helps us to figure out the amount of sticks and spheres with only the size of the tetrahedron given. In the end, every group got the same *n-th* term for the number of sticks, which was $U_n = 6 \times 4^{(n-1)}$ (*n* is a variable for the size, meaning you have to substitute *n* with the tetrahedron size); however, the *n*-th term for the spheres varied from group to group.

1. First, we started with the base formula of every geometric sequence
   This was: $U_n = U_1 \times r^{n-1}$
   U_1 = first term
   r = the ratio between terms (ie. $\frac{24}{6} = 4$ & $\frac{96}{24} = 4$)
2. Our formula thus becomes: $U_n = 6 \times 4^{n-1}$

*All the formula answers match the number of sticks and it uses the variable **n**, meaning it works.*

Size (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**
# of Spheres	4	10	34	*130*	*514*	*2050*	*8194*
# of Sticks	6	24	96	*384*	*1536*	*6144*	*24576*
$U_n = 6 \times 4^{(n-1)}$	6	24	96	384	1536	6144	24576

The n-th Term: Spheres

There were 3 formulas generated by the class regarding the number of spheres.

The first one made was $U_n = 4^{(n-1)} - 6$.

Size (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**
# of Spheres	4	10	34	*130*	*514*	*2050*	*8194*
# of Sticks	6	24	96	*384*	*1536*	*6144*	*24576*
$U_n = 4^{(n-1)}-6$	**-5**	**-2**	**10**	**58**	**250**	**1018**	**4090**

Here you can see that the answers from the formula do not match the number of spheres, therefore it is wrong.

The next formula was (b - 3) x 4 + 6

b is the previous number, so when looking for the number of spheres in size 3, b = 10.

This expression works, but you can't find the n-th term with it.

Size (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**
# of Spheres	4	10	34	*130*	*514*	*2050*	*8194*
# of Sticks	6	24	96	*384*	*1536*	*6144*	*24576*
(b-3) x 4+6		**10**	**34**	**130**	**514**	**2050**	**8194**

Although the calculations are correct for this formula, you cannot calculate the first size and since it can not use n, which is our goal.

With some help from our teachers and the previous UngeAbel winners, we discovered another formula which seemed to work perfectly well: $4 + (6 \times 4^{n-1}) - (2^{2n}+2)$. Something new with this formula is how it corresponds to the stick formula. However, we cannot use the base geometric formula to find the n-th term for the spheres because the ratios in between the numbers are different, however, we did use aspects of it such as the first term in the sequence.

Size (n)	1	2	3	4	5
# of Sticks	6	24	96	*384*	*1536*
$6x4^{n-1}$	6	24	96	384	1536
# of spheres = # of sticks + # of spheres in the previous size (6 + 4 = 10)					
# of Spheres	**4**	**10**	**34**	***130***	***514***
$4 + (6 \times 4^{n-1})$	10	28	100	388	1540
We need to add 4 since it is the first number of the spheres sequence					
Difference	6	18	66	258	1026
This is the difference between # of spheres and current formula (10 - 4 = 6)					
-2	4	16	64	256	1024
Discovery: the difference -2 = every other binary number, same pattern as # of spheres					
2^{2n}	4	16	64	256	1024
Formula to make every other binary number					
$2^{2n} + 2$	6	18	66	258	1026
Formula to make every other binary number +2.					
$4 + (6 \times 4^{n-1})-(2^{2n} + 2)$	**4**	**10**	**34**	***130***	***514***
Here is our final formula. It matches the # of spheres, therefore, it works					

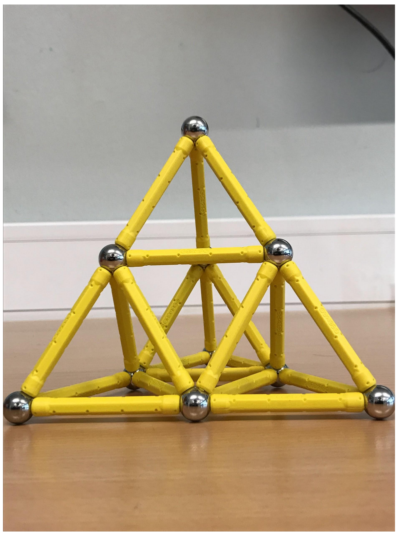
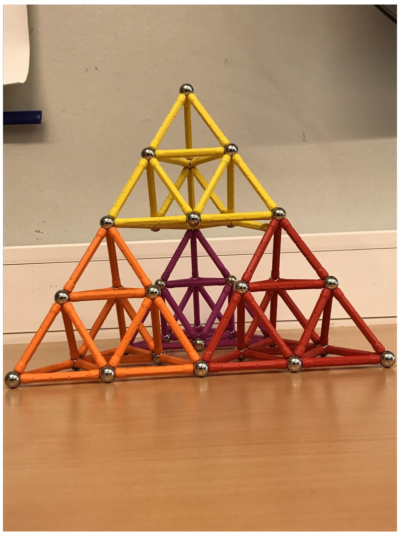
Different Representations of the mathematics

Representation one: Sticks and spheres formula

The number of spheres in the next size can be figured out by adding the number of sticks and spheres from the previous size. For example, to figure out how many spheres there are in size two, one adds the number of spheres (four) and the number of sticks (six) to get ten, which is the number of spheres in size two.

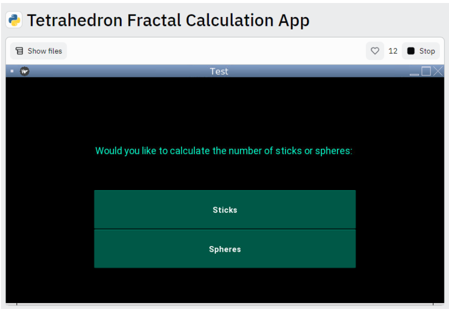
Representation two: Patterns found in the physical construction

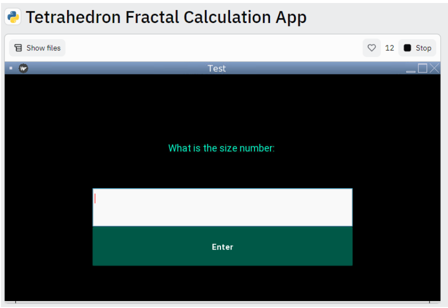
The number of sticks multiplies by four for each size. For example, from size one to size two you multiply the number of sticks in size one (6) by four, you then get 24, which is the number of sticks in size four. You can see that three more of the previous size make the next size in the construction in the next stages:

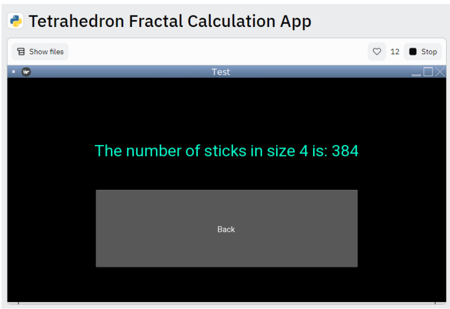




Representation three: Python and L-systems (Repl.it Turtle)

We made a digital code on Python that uses the tetrahedron's size to determine the number of spheres and sticks in the size mentioned[1].

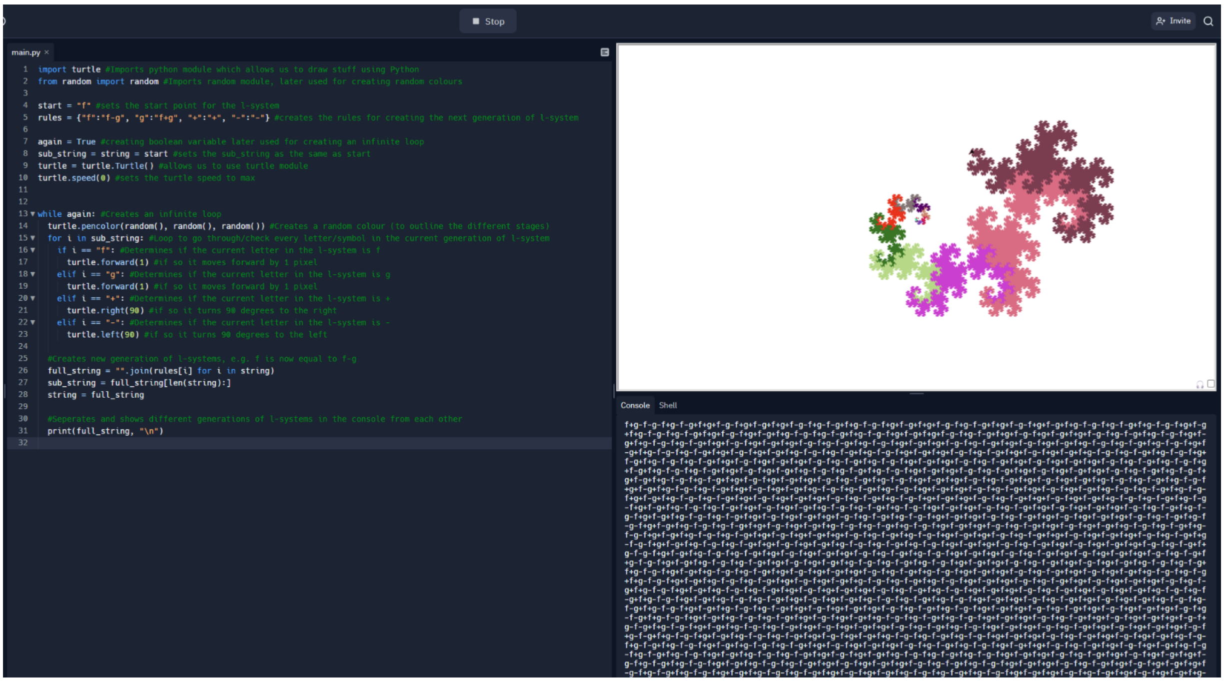


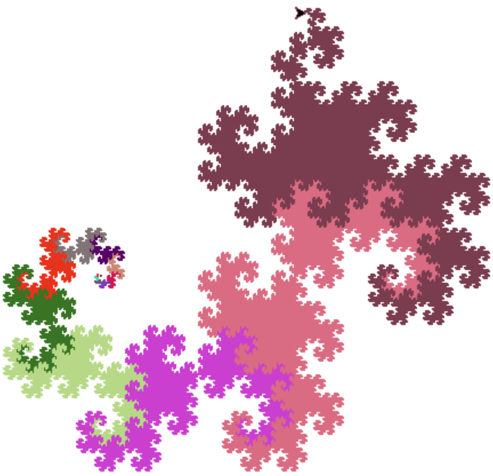




We also generated a digital fractal code, called the Dragon curve fractal, using the turtle module on Python[2].

Each new colour represents a new stage of the fractal.





To make this fractal we used an 1-system. A 1-system is a set of rules that describe how to convert a string into instructions which create patterns/fractals that then develop into new strings based on a set of rules, e.g. x → xy and y → x. This means that if our starting point is *x* our next generation would be x*y* → *xyx* → *xyxxy* → *xyxxyxyx* → *xyxxyxyxxyxxy* …

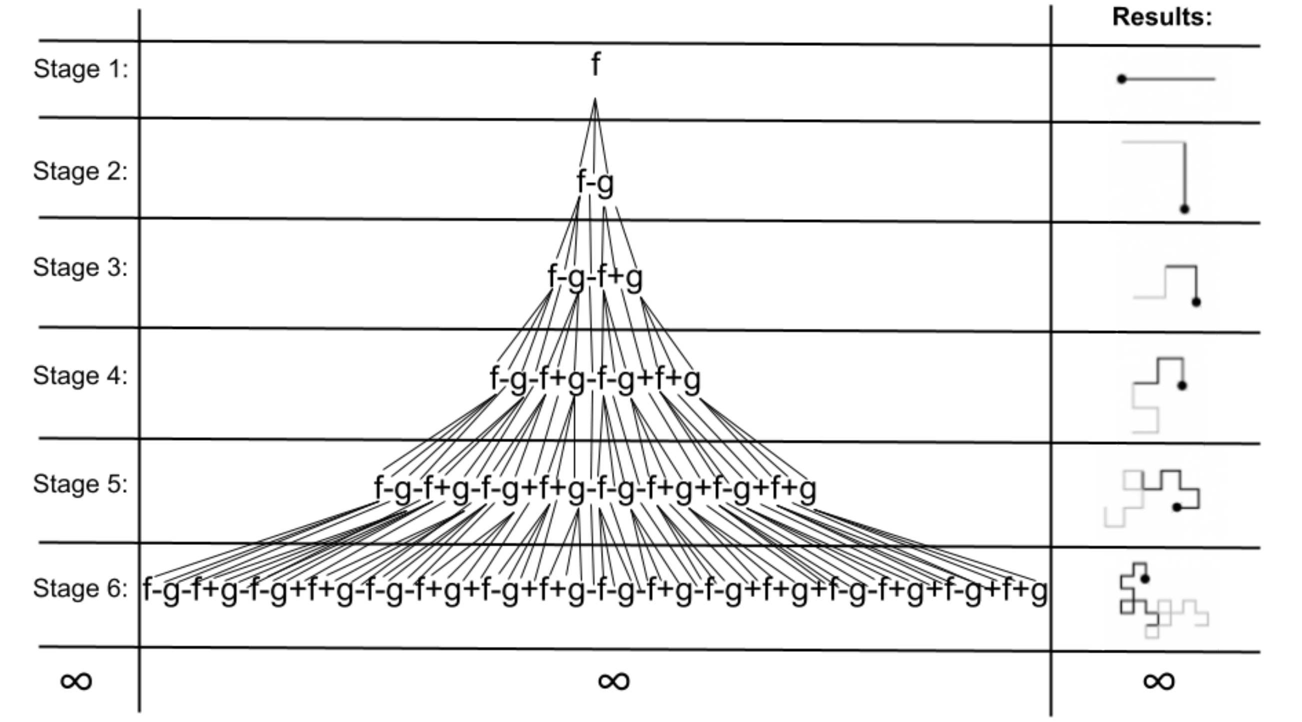
To turn this set of letters/symbols into a fractal, you must add a set of instructions. For example, every time you encounter an x you move forward, and every time you encounter a *y* you turn right 90^0. The computer program first runs through the first generation and follows the instructions, then makes the next generation of the l-system using the predefined rules, then follows that generation of the l-system using the instructions to continue drawing a pattern. This process repeats infinitely and in turn gives a fractal depending on the l-system.

Here is our l-system:

L-Systems: Dragon Curve Fractal
Essentially, the python program has the same rules and instructions coded into them. Everytime the python program runs through a new generation/stage, it draws it using Python Turtle.

Rules:
f = f - g
g = f + g
\+ = +
\- = -

Instructions:
f = forward
g = forward
\+ = right 90°
\- = left 90°



```python
import turtle #Imports python module which allows us to draw stuff using Python
from random import random #Imports random module, later used for creating random colours

start = "f" #sets the start point for the l-system
rules = {"f":"f-g", "g":"f+g", "+":"+", "-":"-"} #creates the rules for creating the next generation of l-system

again = True #creating boolean variable later used for creating an infinite loop
sub_string = string = start #sets the sub_string as the same as start
turtle = turtle.Turtle() #allows us to use turtle module
turtle.speed(0) #sets the turtle speed to max


while again: #Creates an infinite loop
  turtle.pencolor(random(), random(), random()) #Creates a random colour (to outline the different stages)
  for i in sub_string: #Loop to go through/check every letter/symbol in the current generation of l-system
    if i == "f": #Determines if the current letter in the l-system is f
      turtle.forward(1) #if so it moves forward by 1 pixel
    elif i == "g": #Determines if the current letter in the l-system is g
      turtle.forward(1) #if so it moves forward by 1 pixel
    elif i == "+": #Determines if the current letter in the l-system is +
      turtle.right(90) #if so it turns 90 degrees to the right
    elif i == "-": #Determines if the current letter in the l-system is -
      turtle.left(90) #if so it turns 90 degrees to the left

  #Creates new generation of l-systems, e.g. f is now equal to f-g
  full_string = "".join(rules[i] for i in string)
  sub_string = full_string[len(string):]
  string = full_string

  #Seperates and shows different generations of l-systems in the console from each other
  print(full_string, "\n")

```

Representation four: Spreadsheets

We used google spreadsheets, a tool we have used previously in maths lessons to represent the data we had previously collected. This helped us find patterns within this mathematical problem.

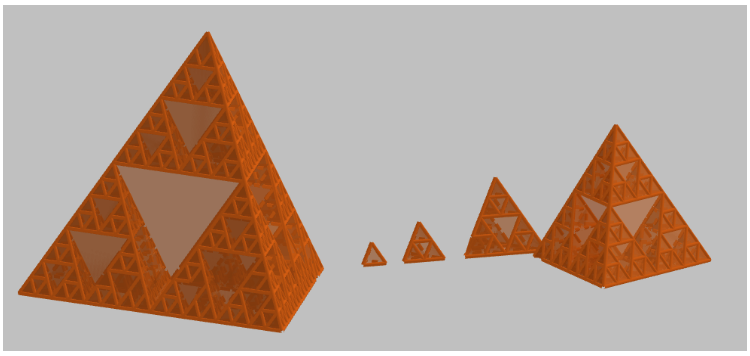
Size(n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Term-to-term rule**
# of spheres	4	10	34	*130*	*514*	*2050*	*8194*	x4-6
# of sticks	6	24	96	*384*	*1536*	*6144*	*24576*	x4
$U_n = 4 + (6 \times 4^{n-1}) - (2^{2n} + 2)$	4	10	34	*130*	*514*	*2050*	*8194*	x4-6
$U_n = 6 \times 4^{(n-1)}$	6	24	96	*384*	*1536*	*6144*	*24576*	x4

*Italicized numbers show the numbers that we found out using the formula

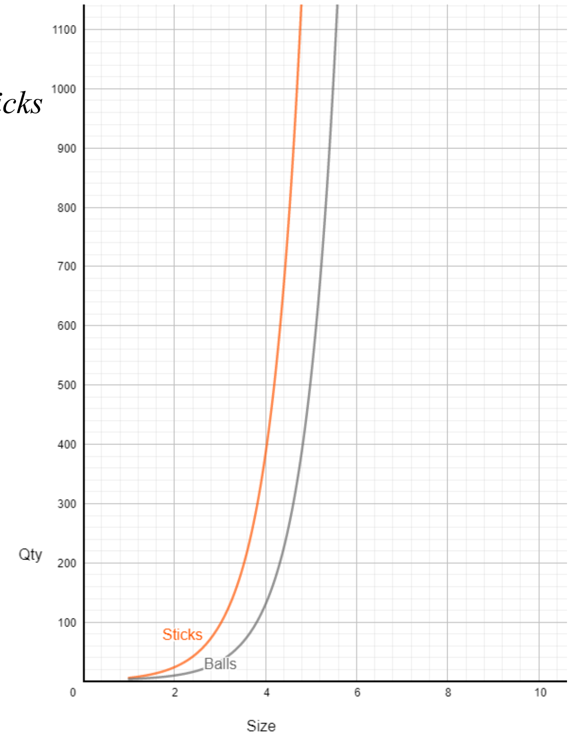
Representation five: Geogebra

For this project, we used Geogebra to represent our findings through images and graphs.

Here is a visual representation of Sierpinski's Triangle



This graph, also made in Geogebra, showed us the exponential increase in sticks and spheres, from size 1 to size 1000



Increase in spheres (blue) and sticks (red)

- Y-axis = number of balls and sticks
- X-axis = Size of shape

Both start at the same point, and the increase is exponential. One more pattern found here is that the difference between the spheres and sticks is constant from quantity approximately till 100 units.

Part 3: Conclusion

The conclusion/solution

This investigation has been a project full of many surprises and uncertainties for both the students and the teachers. Although there was a lot of confusion, delay and inefficiency, a lot of effective learning occurred and it rapidly increased our knowledge of fractals. We split our tasks into groups, and that made it a lot easier to come up with good ideas and methods. We started thinking more about where common fractals are, how they are formed and who made them.

Reflections of our project

This project has significantly increased our knowledge of fractals, where they are located, how they are made and their contribution to numerous global discoveries. In addition, our knowledge of deriving formulas has become greater, especially finding the *n*-th term. The formulas that we derived ($(4 + (6 \times 4^{n-1}) - (2^{2n} + 2)$ for the spheres, and $U_n = 6 \times 4^{(n-1)}$ for the sticks) showed us that repetition in fractals can be written as equations and then be made digitally, when larger numbers are involved. Furthermore, this project has increased our collaboration skills, adaptability and efficiency.

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