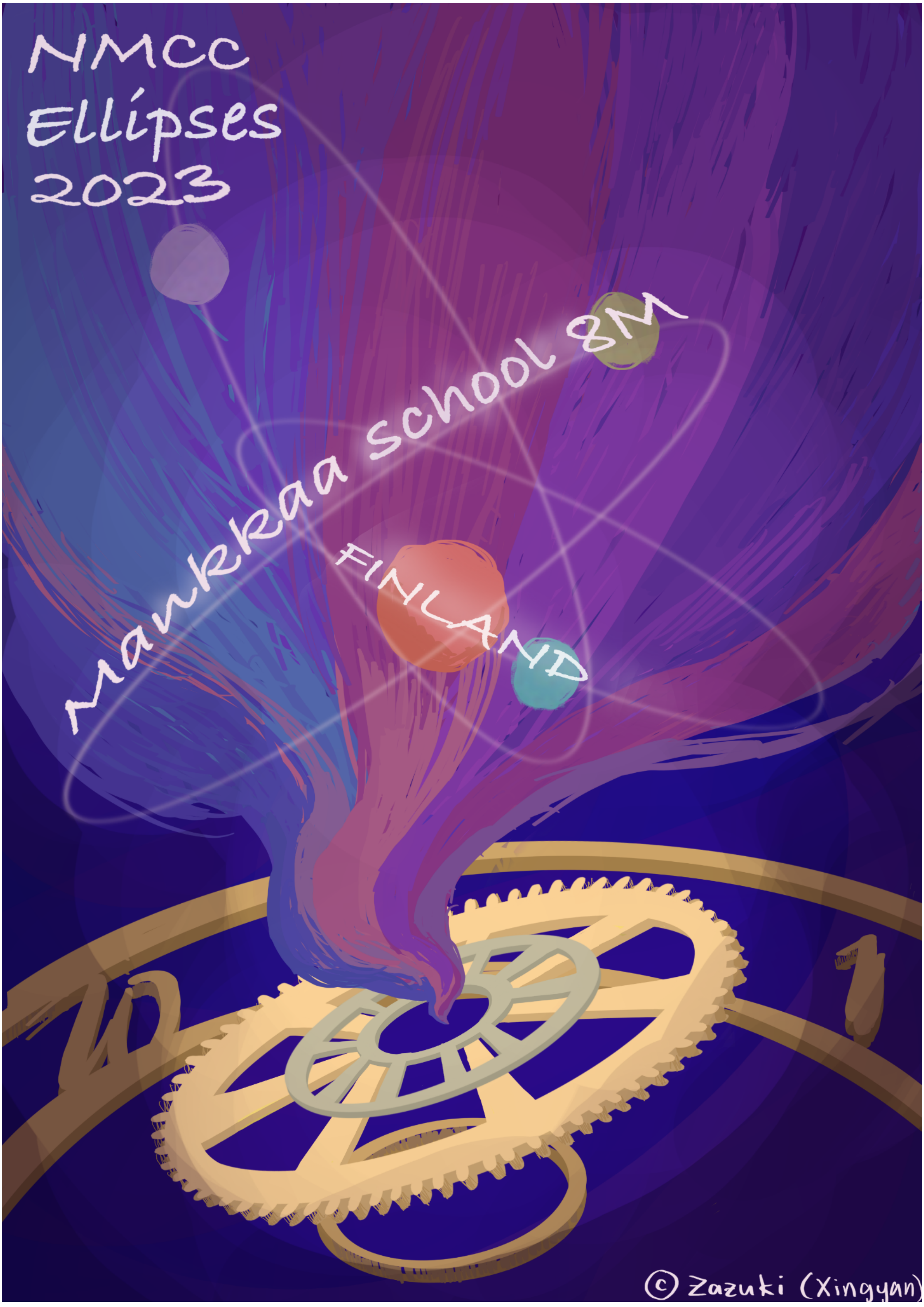
NMCC
Ellipses
2023
Mankkaa school 8M
FINLAND
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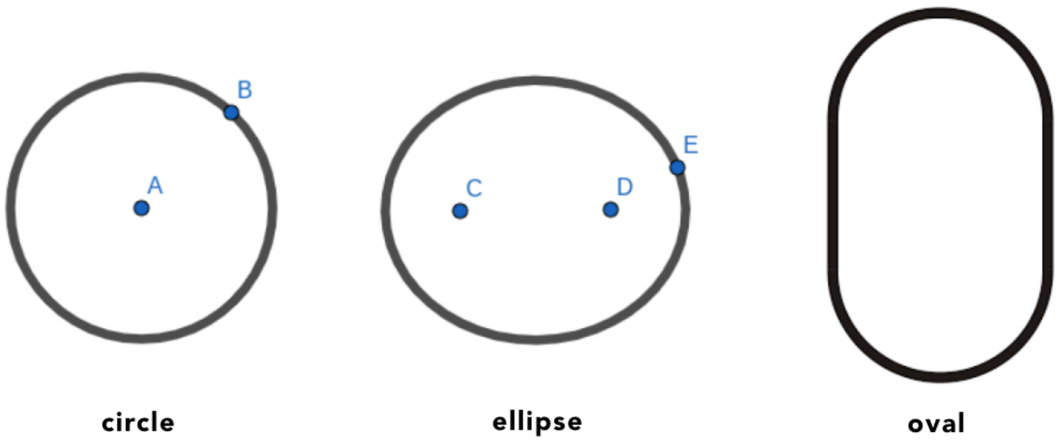
Content

1. Introduction



We studied ellipses as a part of the NMCC competition. In our opinion, ellipses were a great subject because at first we knew mostly nothing about them.

An ellipse is a set of points made of focal rays from the focuses.

Every ellipse is an oval, but not every oval is an ellipse. A circle is also an ellipse, only with its focuses on top of each other.

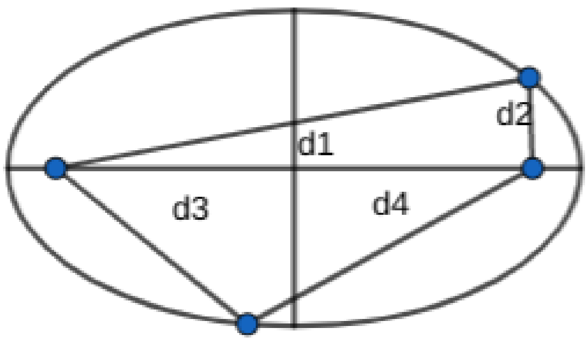
In this project, we will tell you about the mathematical side of ellipses and their relation to the conic section and eccentricity. We also talk about drawing an ellipse.

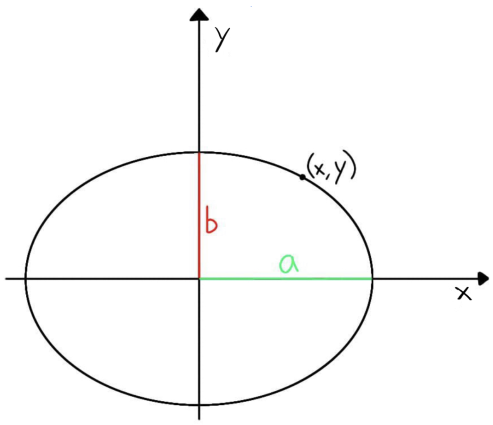
Working on this project has been interesting, since we've never done anything like this together as a class. At first, nobody knew how to start. Our goals seemed very difficult and we understood that we had to discuss things as a group and really concentrate if we wished to achieve them. We encountered many challenges and occasionally got stuck, but despite everything, we survived these challenges and emerged victorious.

2. Definition

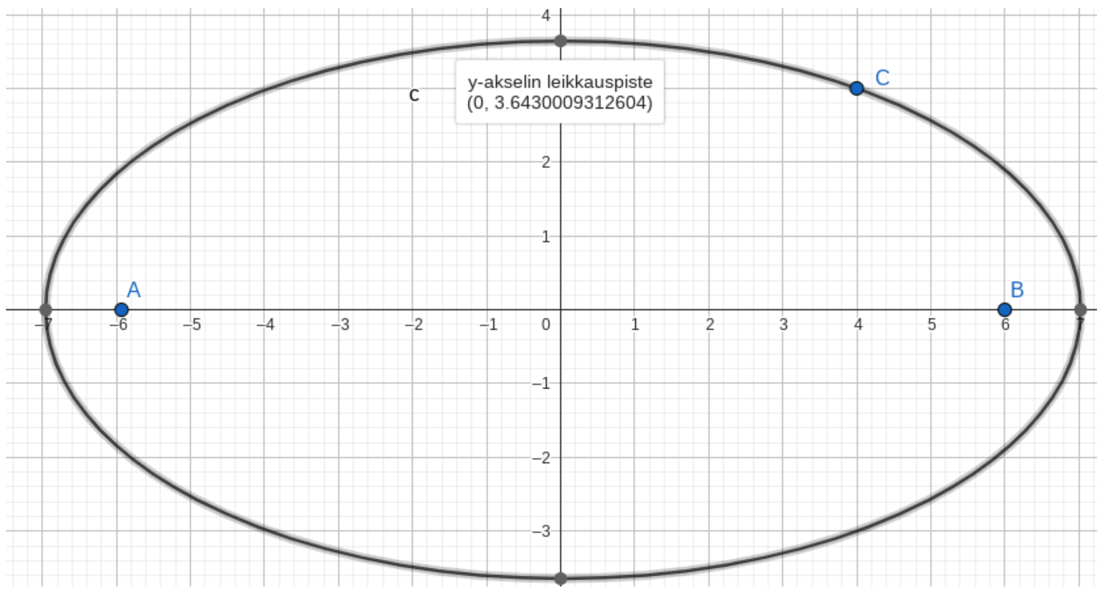
The location of an ellipse's two focuses defines the form of the ellipse along with its focal rays. The location of the focuses can define if the major axels $2a$ and the minor axels $2b$ are known. The sum of the focal rays is constant. The focal rays are the lines that go from the focuses to the perimeter.

This means that $d_1 + d_2 = d_3 + d_4$





In the coordinate system, the origin-centered ellipse's every point has a formula that applies to them. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In this formula a is the major axel's half and b is the minor axel's half. x and y indicate the location of the on the perimeter of the ellipse.

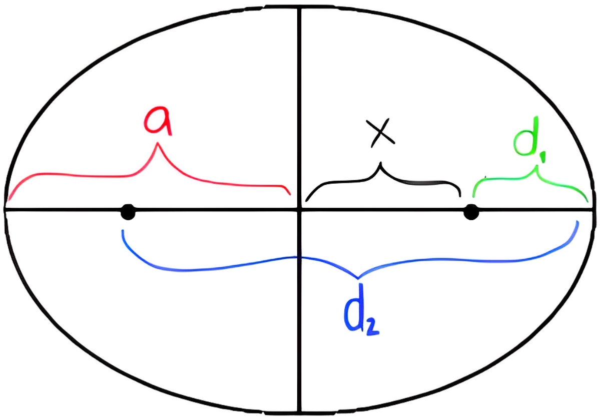


We made an ellipse in Geogebra and we placed its values in a formula. $\frac{4^2}{7^2} + \frac{3^2}{3.64^2} = x$.

The answer was $x = 1$, so the formula was correct.

2.1 Focuses

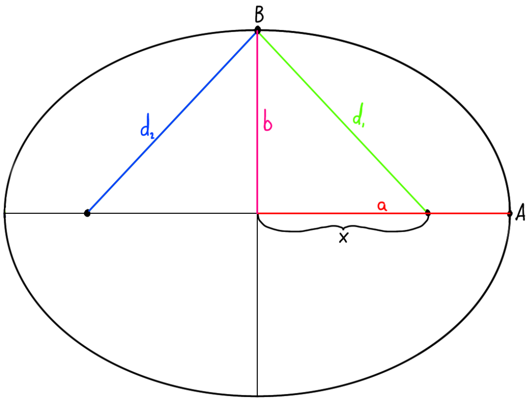
If the axis of the ellipse is known, you can calculate the location of the focuses with two different special cases.



$d_1 + d_2 = (a - x) + (a + x) = a - x + a + x = 2a$

This formula describes a situation where the focal rays are on the major axel but since $d_1 + d_2$ is always constant by the definition of the ellipse, the result $d_1 + d_2 = 2a$ applies generally.

In the second case $d_1 = d_2$, so d_1 and d_2 go directly from the focal point to the intersection of the minor axel and the ellipse's perimeter.



Because the triangle is rectangular and the two sides' lengths are known, we calculated the location of the focuses with the Pythagorean theorem.

$d^2 = b^2 + x^2$

$d^2 - b^2 = x^2$

$x = \pm \sqrt{d^2 - b^2}$

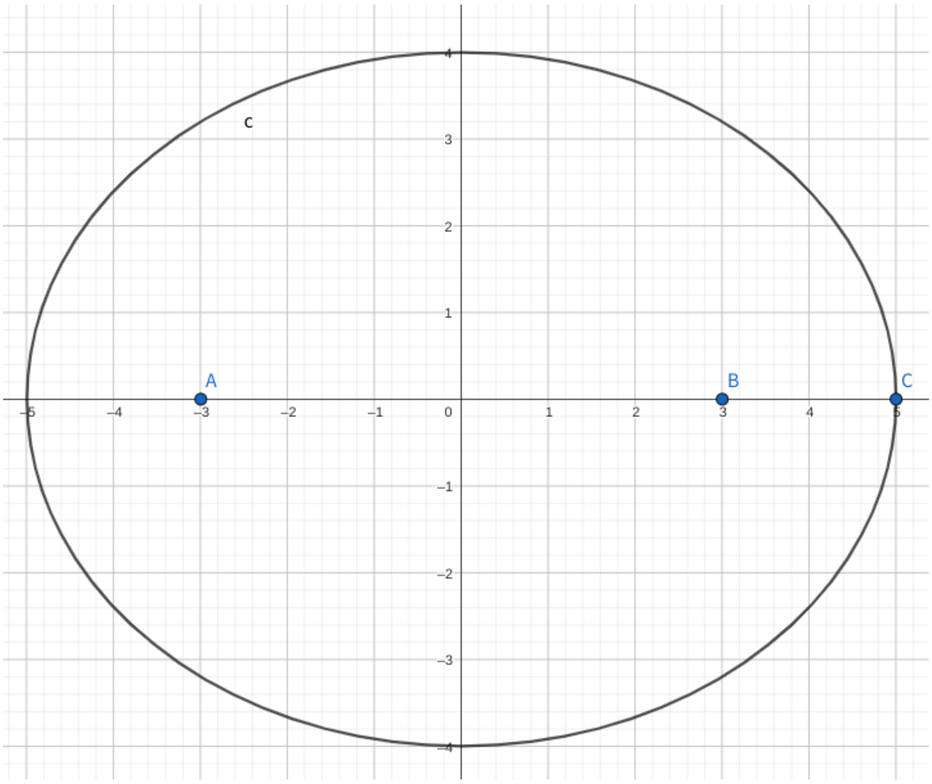
Previously we proved that $2d = 2a$, so $d = a$.

So $x = \pm \sqrt{a^2 - b^2}$ and the location of the focuses can be obtained using the lengths of the half axels.

We used an example to check it where $a = 5$ and $b = 4$ and we calculated where the focuses would be in the coordinate system.

$x = \pm \sqrt{a^2 - b^2} = \pm \sqrt{5^2 - 4^2} = \pm \sqrt{25 - 16} = \pm \sqrt{9} = \pm\ 3$

The focuses are in the coordinate system (3,0) ja (-3,0)

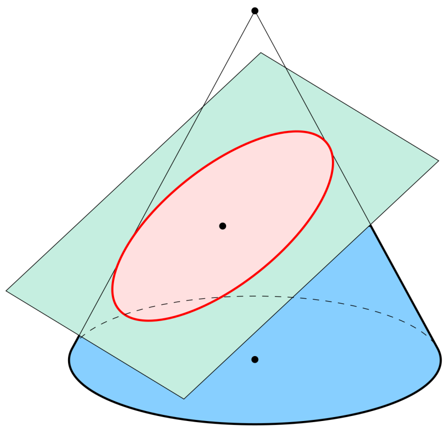


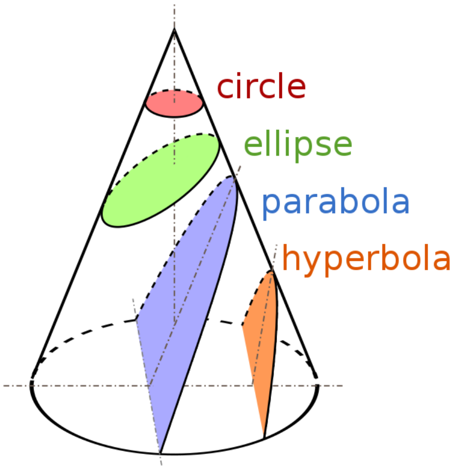
2.2 Conic section

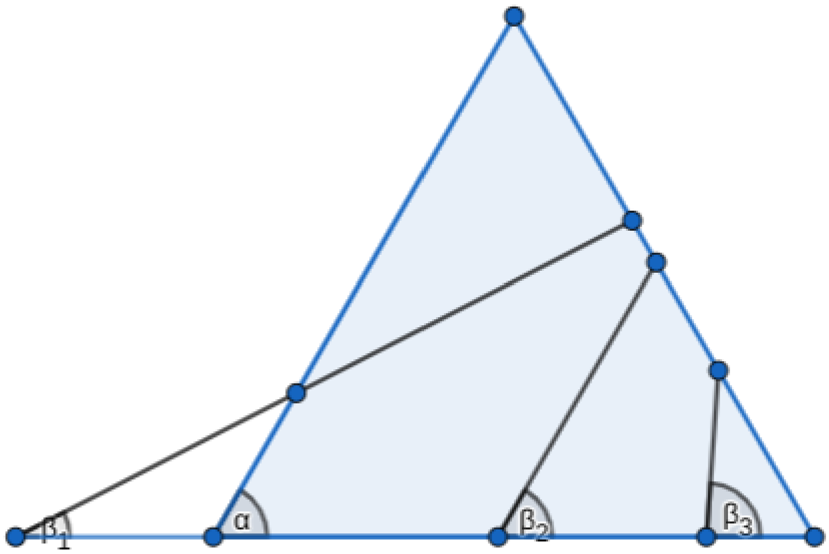
An ellipse is formed when a cone is cut on a slant without cutting the bottom of the cone. Ellipse is just one in four possible conic sections. In addition to the ellipse there are circle, parabola and hyperbola.

Emil and Milo started designing an animation of the conic section well in advance, and they really worked on that for a while.

A circle is formed when the cut is parallel to the bottom of the cone.







If the angle on the cutting plane is wider than the angle of angle a, then it will generate a hyperbola.

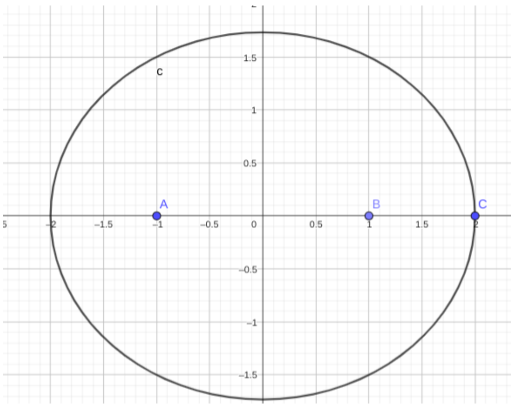
If it is as wide as the angle a, it generates a parabola. And if the cutting angle is smaller than the angle a, it will generate an ellipse. If the angle of inclination is 0° compared to the angle a, a circle will be formed.

2.3 Eccentricity

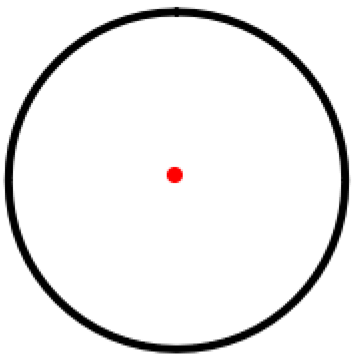
The eccentricity of an ellipse describes how flat or round it is. As it gets closer to zero, it grows more circular; and as it gets closer to 1, it gets flatter. The eccentricity of an ellipse is always between one and zero.

Eccentricity is marked as e and $e = \frac{x}{a}$, where x is the length between the focuses and a is the length of the major axel.

$e = 1$



$e = 0,5$



$e = 0$

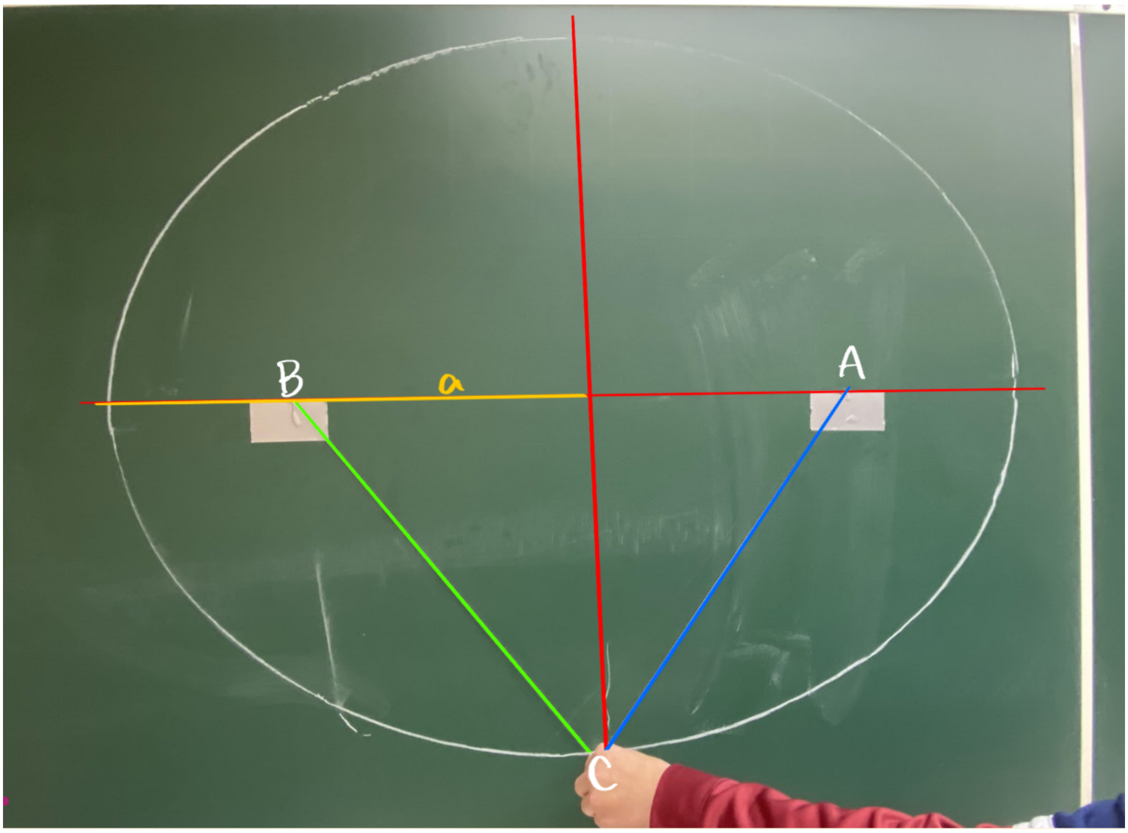
3. Formation

You can form ellipses in many different ways. We became familiar in these ways in different parts of the project, e.g with Geogebra and drawing on paper. Here are a few of the ways we tried and succeeded.

3.1 String

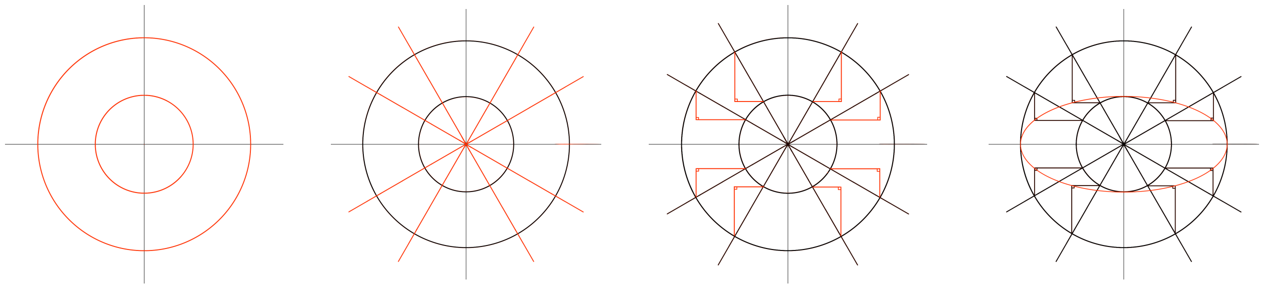
If you want to draw an ellipse without any special tools, it can be done easily with only a string and two pinpoints. First, attach the ends of the strings to the focal points. Then, place the pen inside the string, and start drawing while the string stays tense and the pen stays close to the string. The points where the strings are attached are the focal points, and the shape is an ellipse by its definition.

Drawing with string like this was the easiest way to draw an ellipse for us, and we made this together with Liisa K, Riina, Tuomas, Kaisla and Liisa S.



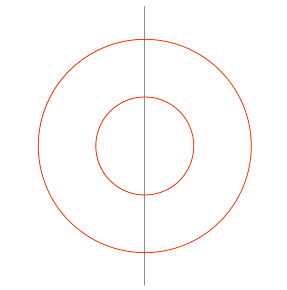
$distance(A, C) + distance\ (B, C) = 2a$, where $2a$ is the sum of the focal rays and two times the length of the major axel. A and B are the focuses of the ellipse and C is the set of ellipses points.

3.2 Concentric circles

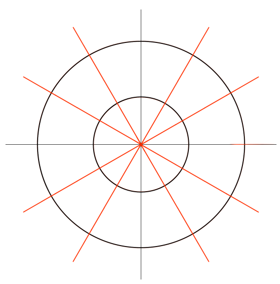


You can also draw an ellipse using two circles. Here we tell you a method where you only need a ruler, a pen and a drawing compass. It's hard to make a perfect ellipse this way, because you would need to do all of the steps an infinite amount of times. At first we were hesitant to try this method as the guide was pretty complicated, but after trying it a few times we confirmed that this method does indeed work.

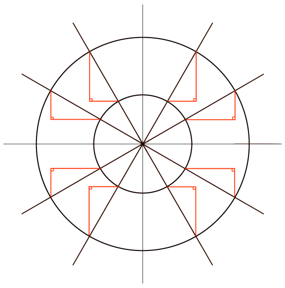
In this method, you draw an ellipse using two circles of different sizes with the same center point.
The radius of the smaller circle should be as long as the forming ellipse’s minor axel and the radius of the bigger circle should be as long as the forming ellipse’s major axel.



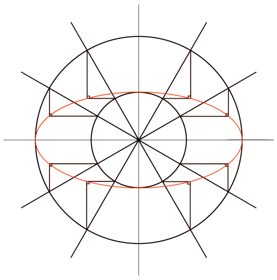
After drawing the circles, you should separate them into smaller same sized sections. In our example, we separated it into 12 sections. The smaller the sections you separate them into, the more accurate the forming ellipse will be.



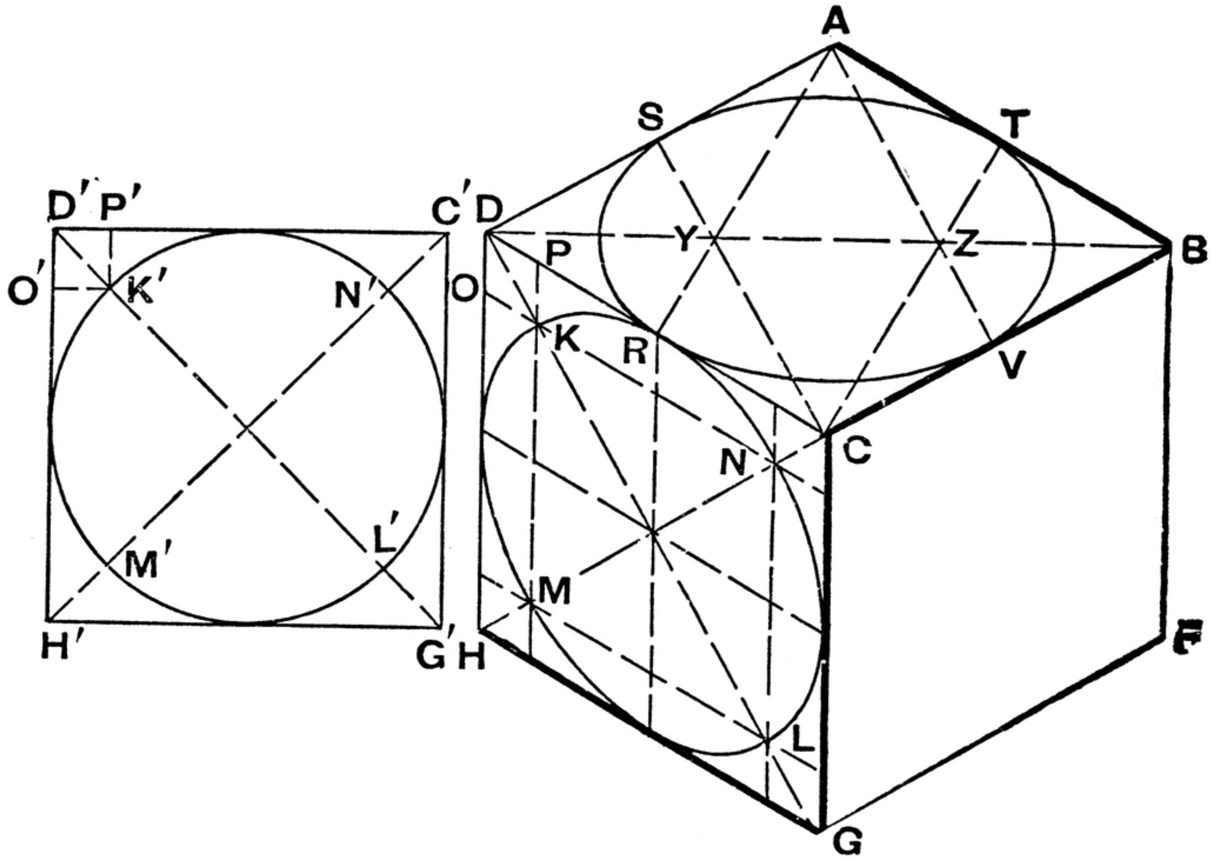
The next step is to draw rectangled triangles. The hypotenuses should be drawn on the line between the two circles. The base should be parallel as the ellipse's major axel, and the sides should be parallel to the minor axel.



The last step is to draw an ellipse by combining the straight corners to each other.



3.3 Perspective



This is the easiest way to make an ellipse and it is based on perspective. You simply need to draw a circle and tilt it. The circle changes into an ellipse, although this can be difficult to see. Apollonius Pergalainen proved it while researching the conic section in 200 B.C., that if a circle is tilted either way, every visible form is an ellipse.

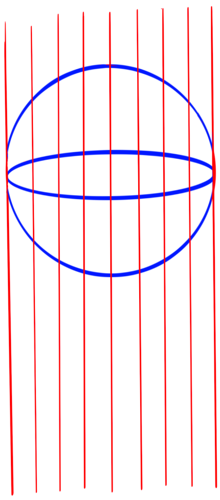
In the picture above there are two ellipses, because the circle is seen in a perspective in which it is slightly flattened.

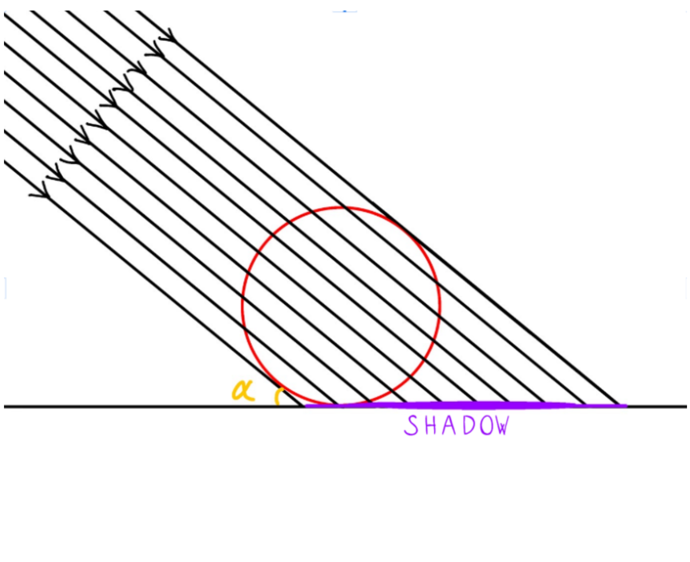
3.4 Shadow

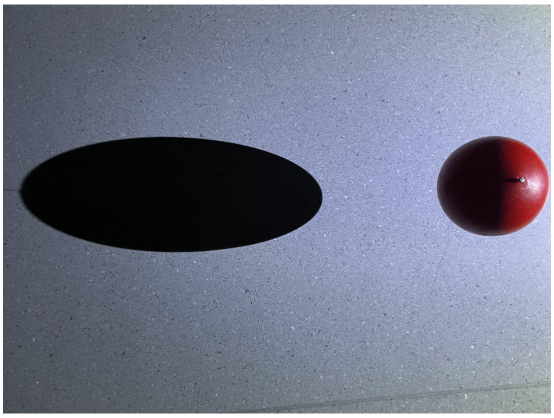
You can also form an ellipse using a flashlight and a ball. When the light shines vertically from the top of the ball, the shadow will look like a circle. When we change the angle of the flashlight, the shadow slowly turns into an ellipse. The clearest ellipse is formed when the light comes from a relatively upper slope.

The shadow makes a 2D figure from a 3D one. So to make a shadow we can just focus on the ball as a circle. The parallel rays of light go in a straight line onto the circle and if perpendicular to the ground will form a circle. This light between the circle and the ground will form a cone that is approaching infinity in height. Then it will work as the conic section, so if you cut it at a certain angle it will form either a circle, an ellipse, a parabola or a hyperbola. The moving angle of the light if looking at the ground will make the shadow's form.

You can also prove it in another way. We will still be focusing on the circle if we can assume that the circle consists of small lines. When there's an infinite amount of lines, the circle is perfect. The light shines on this circle in straight lines and when the shadow appears on the surface, the lines will be deformed into longer ones or stay the same. The angle of the light defines how long the shadow will be and how the lines will be deformed. The longest line is the one that has an angle of 90° to the ground and the lines that will stay the same are either on the circles top or bottom, since their angle is 0° to the floor.

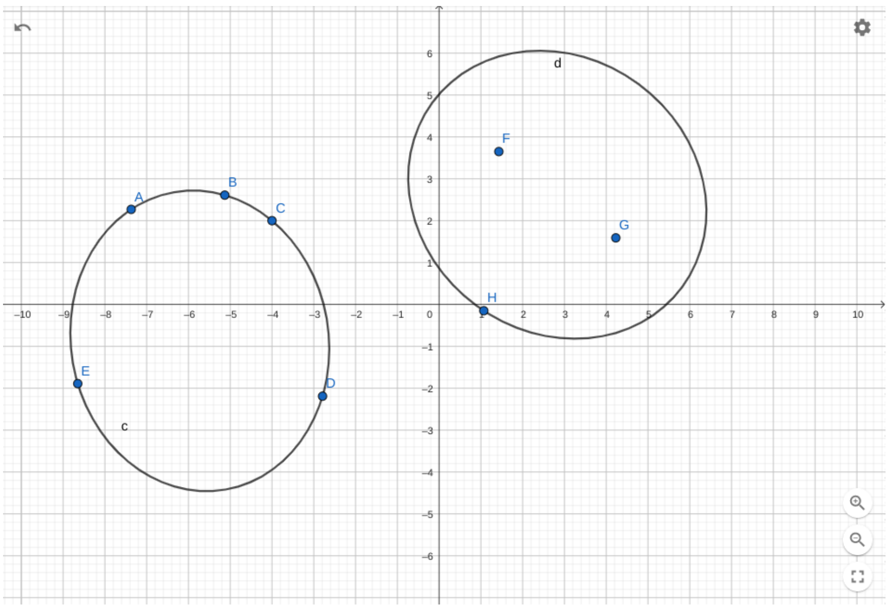






In our opinion, the most visually appealing ellipse is created this way, as shown in the picture.

3.5 Computer



If you use a computer, use Geogebra where there are two methods: using the focuses and a point on the perimeter or making an ellipse with five points on the perimeter.

3.6 Comparison

Part of the ellipses can be made more easily than others, like the circle which you can just tilt to make an ellipse, otherwise, the ellipse made with the string, is a bit more difficult, more demanding, and requires more accessories. There are also some differences to make ellipses. The ellipse made with the shadow is an ellipse only if the object is a ball. The ellipse made using a computer is perfect. If you make an ellipse using a string, it is very difficult to make a flawless one, since the string's tension can change easily. Neither is the method with concentric circles perfect since the points need to be drawn an infinite amount, which is impossible.

4. Overview

Our mission was to answer questions regarding ellipses and ponder the concept of ellipses as a whole. At first we didn’t know what challenges laid before us, but we knew that we could overcome all hardships if we worked as a group and utilized everyone’s strengths.
Our class has all kinds of people; from gifted mathematicians to skillful writers.
The project’s cover was drawn by Zaza with her amazing art skills, and our various methods of drawing ellipses were tested by Aleksanteri, Joonas, Tianye, Teodor, Oskari and Ville. Most of the math was figured out by Viljami, Aaro, Roope and Valtteri. Nunu, Nea and many more worked as a jack-of-all-trades, offering general help to anyone in need.

When we encountered problems, we sorted them out together and continued.
Sometimes we had a lot of trouble, sometimes we argued and disagreed on stupid things. Our biggest challenge was starting and figuring out what to do. We created a long list of ideas, from which we chose the best of the best.

There was also a point in time where a lot of our class couldn’t come to school. We had no motivation to continue it. Yet, we didn’t give up.

All of our versions had many mistakes, and there were always things to fix and improve. But even that didn’t discourage us.

We regularly went through the project with our teachers and carefully worked on any and all errors they found. We got many new ideas and slowly made our project better and better. We removed all unnecessary pieces of text and replaced it with better writing. This project you are reading right now, is our final version - our masterpiece.

5. Photos


NASA
GANT
TROBBS
Original

6. Sources

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Pictures

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