Bilingual Montessori School of Lund

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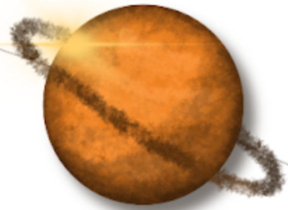
Region: Lund, Skåne
Nation: Sweden
School: Bilingual Montessori School of Lund (BMSL)
Class: 8B



Ellipses

NMCC - sigma 8

CONTENTS

Introduction:

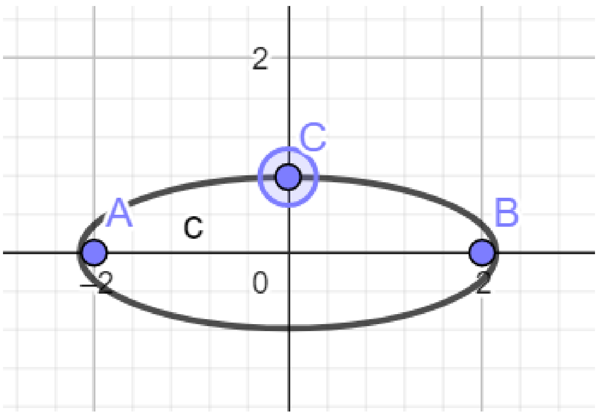
An ellipse is a geometric circle-like shape, which can be created using different methods. We will be presenting a few methods, and mathematically prove why these converge into the same particular shape. Our class has collaborated through challenges and reached intriguing results about the nature of ellipses.

Collaboration & Challenges:

The very first thing we did as a class was brainstorm what we already knew about ellipses. As it turned out we had different ideas about what an ellipse was, for example we believed that an oval was an ellipse, but this is not necessarily the case. An ellipse is a particular type of oval.

We split into five groups of four, where every group was to research the properties of the ellipse, to reach a correct and cohesive idea of what they are. The observations made by the groups were then compiled into a single document. The properties of an ellipse were described as follows:

This ellipse was created in Geogebra, a program specifically chosen as it clearly displays the



properties of an ellipse. Points A and B are the 'focal points' of the ellipse, and their added distance to any point on the side of the ellipse is constant. The image also shows, the 'axes of symmetry' (the lines slicing the ellips through its centre along the x and y axes). These are found in every ellipse, and part the shape into four congruent (equal) parts. They slice each other perpendicularly and are called the major axis and the minor axis.

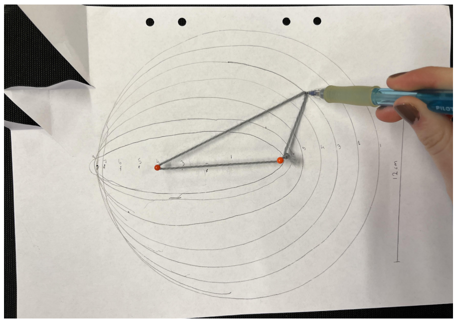
Every ellipse has an inherent value of eccentricity. This value is between 0 and 1 depending on how circular the ellipse is. Given an eccentricity of 0, the ellipse is a circle, for a value of

1, a parabola. Eccentricity can be described as the degree to which an ellipse is distorted in a specific direction, on a scale from 0 to 1.

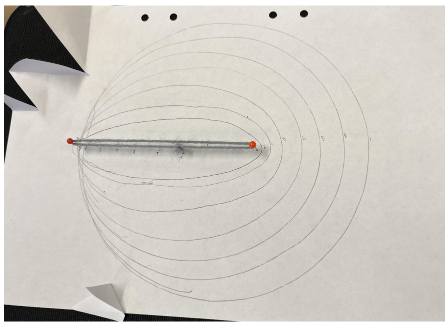
Ellipses have been proven to be particularly significant in the field of optics, since a beam of light passing through one focal point of an ellipse, will always be reflected through the other focal point, no matter the angle of entry.

This is how the different groups collaborated and solved problems:

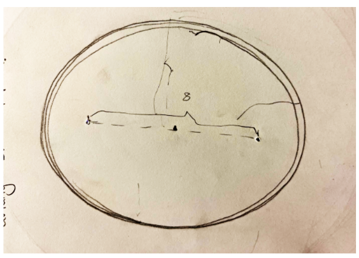
Group 1: After researching, we decided to use the string method, as we found it to be the most interesting and practical in the classroom. We acquired two pins and a string, after which we attached the pins to a flat surface and tied the strings around them. We placed a pencil within the string and drew a seamless line at the furthest point from the focal points the elasticity of the string allowed for. Given the true statement that the string is a constant length, the only shape that can be created is an ellipse.



We found it difficult to understand that the shape of an ellipse is affected by the distance between its focal points. We began by placing the focal points quite close together, creating what closely resembled a circle. After close consideration, we realised that the pins were representative of the focal points of an ellipse. The further the distance between them, the more elongated it becomes, thus solving our problem.



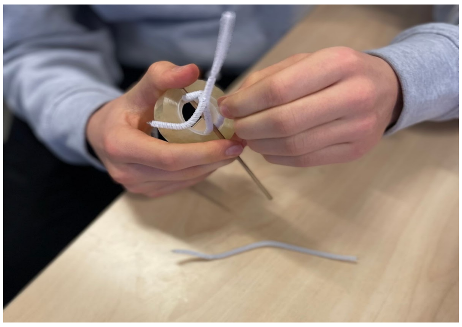
Group 2: Having researched what an ellipse is, and what properties it has, we scoured the internet for how it physically was to be recreated. We considered several methods but took a liking to one method in particular, the string method (explained by the previous group). However, we experienced minor problems with having the pins stay still and in place.



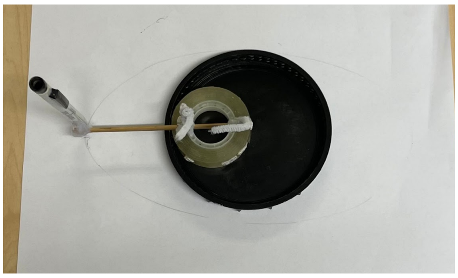
We solved this problem by holding while drawing. The foam rubber casing of our computers were used to place the pins as it provided the rigidity we needed to keep the pins in place.

Together we solved our problem and concluded that the string method was in fact, effective.

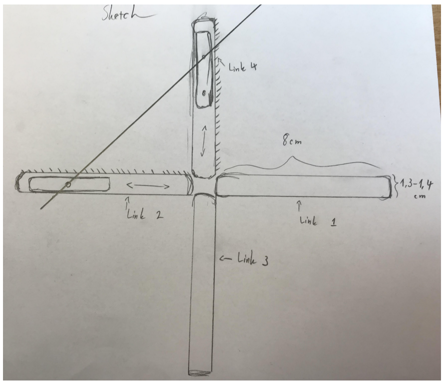
Group 3: Seeing group 4 create a Trammel of Archimedes, we were greatly inspired and decided to create a simplified version that could be replicated in our classroom. In our research, we came across an alternative approach to the trammel, the tusi couple, using a circle rolling within the circumference of a circle with twice the diameter. We decided to use a pencil, a wooden skewer, two pipe cleaners, a lid and a roll of tape. To find a circle with exactly double the diameter of the other was a challenge, but an adequate approximation was found in a lid from the school kitchen. Using a glue gun, we connected the components.

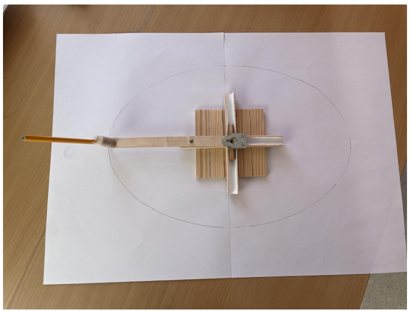


Having put all the parts into place and tested it, we found it very difficult to use. The stick kept slipping so the pencil moved out of position. Adding a small amount of ‘blu-tack’ (adhesive), solved the problem. Additionally, we noticed that the inner circle slipped too much. After several attempts we tried adding ‘blu-tack’ on the inside of the outer circle to increase the friction. This solved the problem since the circles stuck to each other for a short moment. With some practice the machine worked excellently.



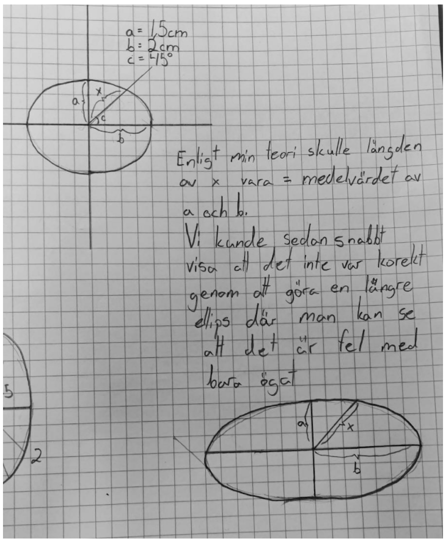
Group 4: It came to us that we had seen a video on tiktok, displaying a mechanism creating an ellipse. We immediately knew we were to use something similar and started searching online. Its name was found out to be; “The trammel of archimedes”. It operates through points moving in sync along straight lines resulting in an ellipse.



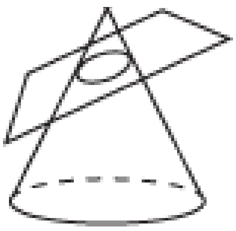


Having understood the basic concept, and the constituent parts in the trammel, we started sketching. We decided for it to be made from wood, plastic rods and metal. At first we did not understand the proportions between the different parts and so our first attempt did not create a true ellipse. We finally created a cohesive sketch with every condition for success. We found our collaboration constructive and efficient, providing us with a successful result.

Group 5: We considered a wide variety of options, one of which is shown on the right. We mathematically proved the string method tested by by several other groups.



We also spoke of another approach to creating an ellipse, slicing a cone, see image to the left. Its essence can be described as follows; A 2-dimensional flat surface intersecting the cone at an angle, without intersecting the base surface of the cone, will always generate an ellipse. Testing this method, we used a transparent cone and a sheet of paper. Cutting an elliptical hole in the sheet, we placed it over the cone, at an angle, and noticed its shape fit the cone perfectly.

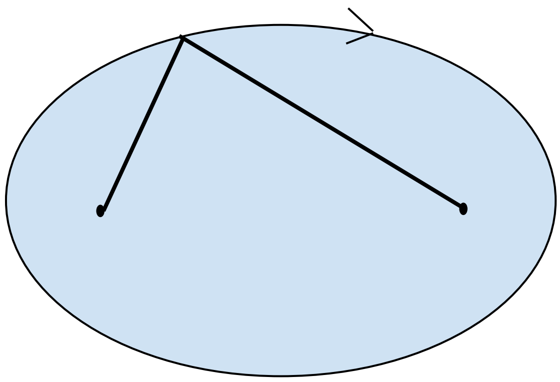


We also wondered if an ellipse could be described with an equation and immediately started investigating this. We assumed that this would be possible, and so we tried to define our measurements. But just as in any other shape, the lengths and proportions of an ellipse can vary, and so the challenge was overcome by differentiating necessary lengths and proportions from unnecessary.

Result:

After going through the experiments made by all the groups, we decided to show why three of the methods create the same shape, an ellipse. We began with the simplest method and progressed to the most difficult.

The String Method:

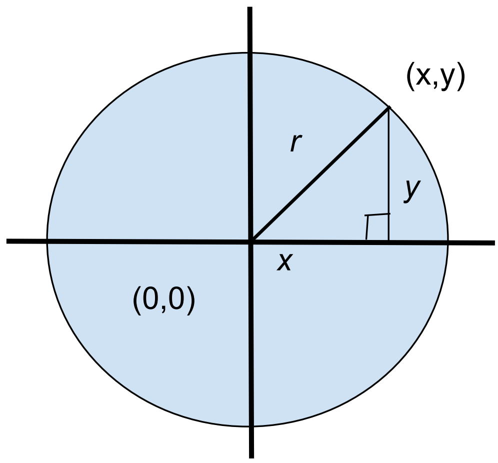


As seen above, two groups have used this method, and we will use it as in defining an elleipse when proving the remaining methods.

Our definition reads as follows: *“A shape is an ellipse if two focal points can be placed such that given any* *point on the edge of the shape, the distance from each focal point to the point on the edge is always the same."*

The String Method and the Equation of an Ellipse:

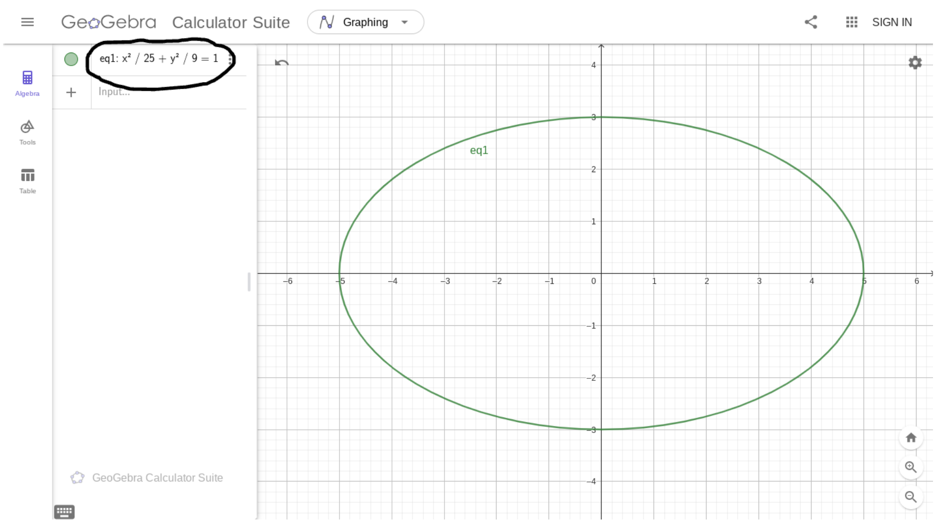
With digital tools such as Geogebra, you can draw circles with the equation of a circle. For a point with coordinates (*x,y*) to be at the edge of a circle with radius r and centred at the origin, the following equation must be true: $x^2 + y^2 = r^2$. We can prove this formula by using the pythagorean theorem. If we would like to draw a circle with radius 5, for example, we could write $x^2 + y^2 = 5^2$.



Can we create a similar equation for an ellipse?

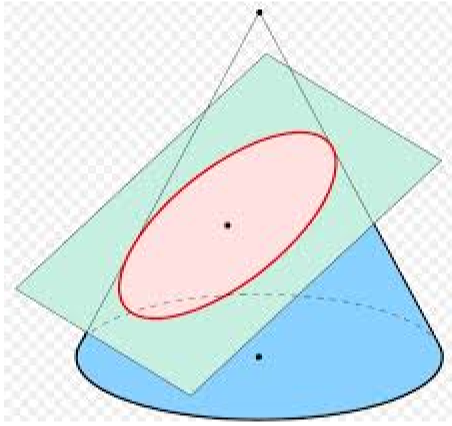
By using our knowledge of the string method and our definition, we can prove that the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a is half the length of the major axis ,and b is half the length of the minor axis and its centre is at the origin.
Appendix 1 shows how we got to this formula.

This formula is the equation of an ellipse. This means that the point (*x,y*) is on the edge of an ellipse with a major axis of 2*a* and minor axis of 2*b*. If we for example would like to draw an ellipse with major axis *10* and minor axis *6,* we could write $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ in Geogebra.



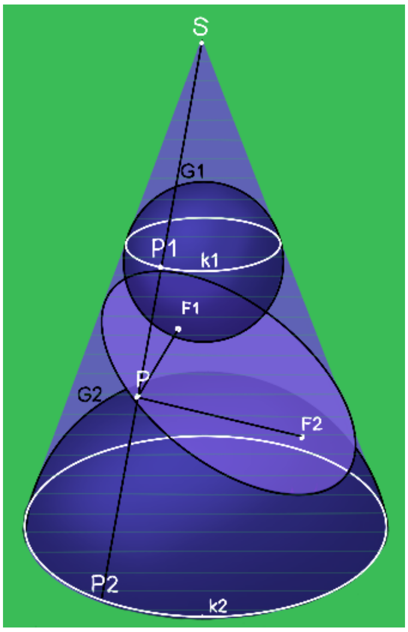
If $a = b$, the equation becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or $x^2 + y^2 = a^2$, the equation of a circle.

This is reasonable, as a circle is an ellipse where both the major axis and the minor axis are of equal lengths.



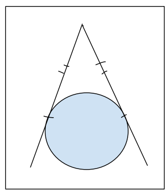
Slicing a Cone:

An ellipse can be created by slicing a two-dimensional plane through a cone at an angle. The cross-sectional surface will then have the shape of an ellipse. This is called a conic section.



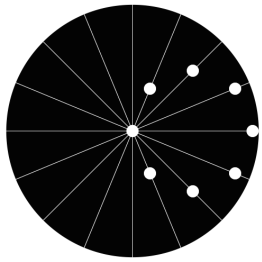
Here is shown why an ellipse can be created in such a way by using the definition of an ellipse created from the string-method.

We start by placing two spheres into the cone (let's call them G1 and G2), G1 is placed above the cross-sectional surface and G2 is placed below. G1 is tangent to the plane in the point F1 and G2 is tangent to the plane in the point F2 (tangent means that a plane or a straight line touches a curved line or surface in a single point). G1 touches the cone in a circle we call k1 and G2 touches the cone in a circle we call k2. The point S is the peak of the cone and the point P is an arbitrary point on the edge of the cross-sectional surface. P1 is the intersection point between circle k1 and the line SP and P2 is the intersection point between circle k2 and the line SP. If two lengths, tangent to the surface of a sphere in one end meet at a given point, the lengths will be equal. The deduction drawn from this is that the distance between P and P2 is equal to the distance between P and F2, and that the distance between P and P1 is equal to the distance between P and F1. This, in turn, means that the sum of distances P to F1 and P to F2 is equal to the distance between P1 to P2. For any given point for P on the edge of the cross-sectional surface, the distance between P1 and P2 is constant, and by extension, the sum of distances P to F1 and P to F2 is constant, indifferently of the position of P. Given our definition of the ellipse, the cross-sectional area is an ellipse with given focal points F1 and F2.

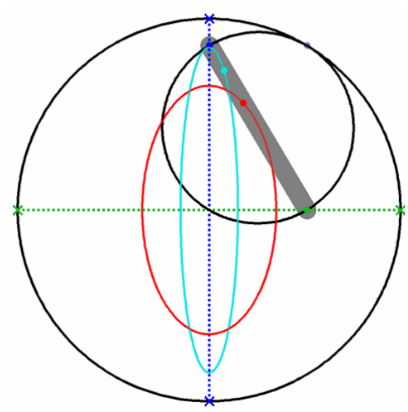


Tusi Couple:

Another means by which to create ellipses is by using the tusi-couple. It consists of two circles, one with any given length, and another with half of the previous circle's diameter, rolling within its inner circumference. During rotation, a specific point on the smaller circle will oscillate, moving only in a straight line.



In this picture, the green and blue dots mark their corresponding green and blue lines which are the axes of the ellipses. The distance between the green and blue points is equal to the diameter of the smaller circle. Given any point along the smaller circle's diameter, or beyond it, its trajectory will create an ellipse. The greater the length, the greater the ellipse becomes.



The picture tells of the congruent nature of both the tusi-couple and archimedes trammel, and the similar means by which these approach the challenge.

Conclusion:

The conclusion made from our research is that the ellipse is a shape that with the help of its focal points can both be created and proven. The sum of the distances between each focal point and any given point on the circumference is constant. An ellipse can be created by slicing a cone at an angle, by means of the string method, the trammel of Archimedes, by finding its equation and using a tusi-couple. Unlike our preconceived idea of the ellipse's geometric nature, they are different to an oval. Additionally, through our research, in some cases more successful than others, we have expanded our knowledge of the ellipse, not simply mathematically, but also within nature and culture.

Sources:

https://youtu.be/MUaWB3XTjxk Drawing an ellipse

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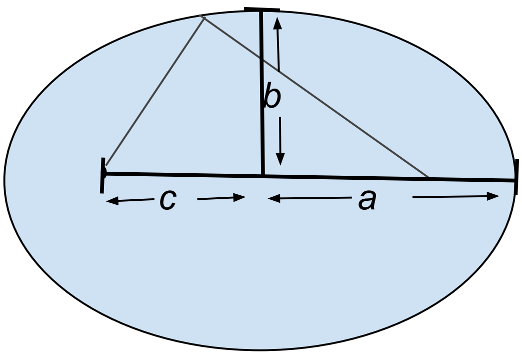
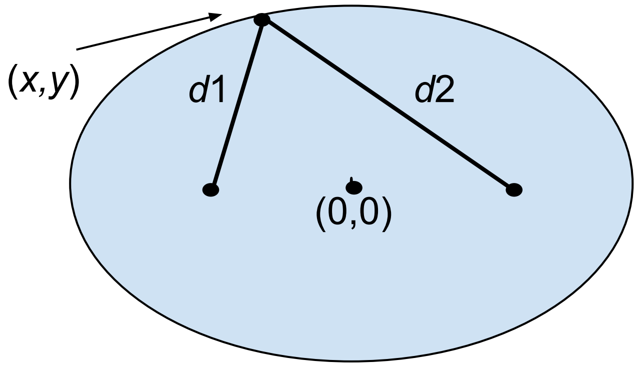
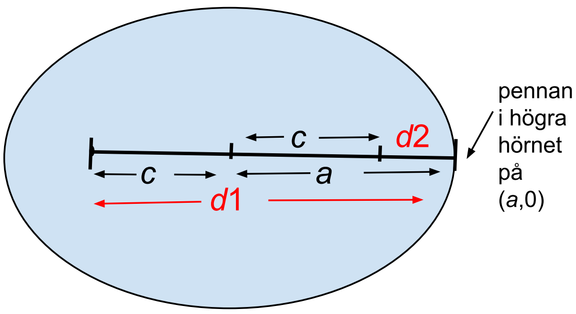
https://blender.stackexchange.com/questions/165190/how-to-do-trammel-of-archimedes-animation' How to do Trammel Of Archimedes Animation?

Appendix 1

Going from the string method and our definition we can create different formulas. We set our a center of an ellipse and label the distance along the major axis a and along the minor axis b. We call the distance from the center to one of our focal points c. The string's length can be divided into 2 parts *d1* och *d2*. The point of the 'pen' has the coordinates (x,y).

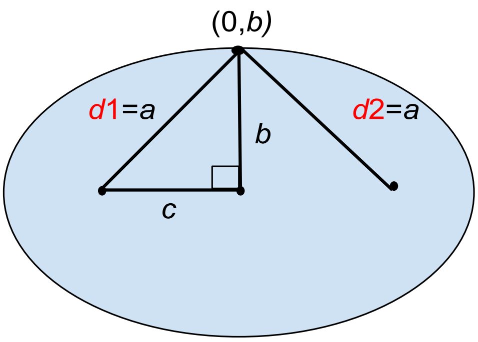






To calculate the length of the string we will take the case where $(x,y) = (a,0)$, that is, when the pen is in the furthest point on the right. $d1$ is then $c+a$ and $d2$ will be a-c. Since the strings length is constant we can write the formula

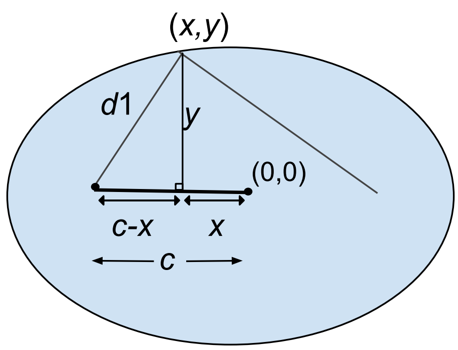
$$d1 + d2 = (c + a) - (a - c) = 2a$$

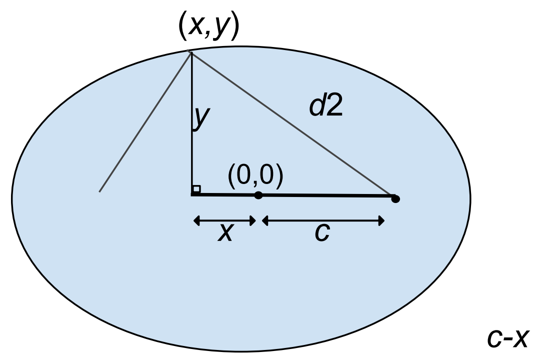


Now we take the case when $(x,y) = (0,b)$, so when the point is in the uppermost position. Since $d1$ och $d2$ are equal then $d1 = d2 = \frac{2a}{2} = a$

This creates a right angled triangle with sides a, b och c. We can use pythagoras theorem to get the formula $c^2 + b^2 = a^2$

Using the pythagorean theorem, the value of lengths $d1$ and $d2$ from coordinates (x,y) are given.





d1=(c-x)2+y2

d2=(c+x)2+y2

Knowing that $d1 + d2 = 2a$, the following equation can be written:

$$\sqrt{(c-x)^2+y^2} + \sqrt{(c+x)^2+y^2} = 2a$$

The coordinate (x,y) is an arbitrary point on the side of the ellipse. We however wish to write this equation with constants a and b instead of a and c. It can be heavily simplified.

$$\sqrt{(c-x)^2+y^2} + \sqrt{(c+x)^2+y^2} = 2a$$

$$\sqrt{(c-x)^2+y^2} = 2a - \sqrt{(c+x)^2+y^2} \qquad \textit{(square)}$$

$$(c-x)^2 + y^2 = 4a^2 - 4a\sqrt{(c+x)^2+y^2} + (c+x)^2 + y^2$$

$$c^2 - 2cx + x^2 = 4a^2 - 4a\sqrt{(c+x)^2+y^2} + c^2 + 2cx + x^2$$

$$-2cx = 4a^2 - 4a\sqrt{(c+x)^2+y^2} + 2cx$$

$$4a\sqrt{(c+x)^2+y^2} = 4a^2 + 4cx$$

$$a\sqrt{(c+x)^2+y^2} = a^2 + cx \qquad \textit{(square)}$$

$$a^2(c^2 + 2cx + x^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

$$a^2c^2 + 2a^2cx + a^2x^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$a^2c^2 + a^2x^2 + a^2y^2 = a^4 + c^2x^2$$ *rearrange the terms*

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$ *factorise*

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$$

We previously created the formula $c^2 + b^2 = a^2$ which can be written as $b^2 = a^2 - c^2$.
Now we can replace all $(a^2 - c^2)$ with b^2 and we are given

$$x^2b^2 + a^2y^2 = a^2b^2$$ *divide by a^2b^2*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This formula gives the equation of an ellipse.